Preliminary Exam in Algebra
Fall, 2015

Instructions:

1. All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.

2. Use separate sheets for the solution of each problem.

Problem 1.
Let G be a finite group such that all Sylow subgroups of G are normal and abelian. Show that G is abelian.

Problem 2.
For a finite group G define the subset $G^2 = \{g^2 : g \in G\} \subset G$. Is it true that G^2 is always a subgroup?

Problem 3.
What is the smallest possible n for which there is an n by n real matrix M which has both:
(a) the rank of M^2 is smaller than the rank of M,
(b) M leaves infinitely many length one vectors fixed.

Problem 4.
Let I denote the ideal in the ring $\mathbb{Z}[x]$ generated by 5 and $x^3 + x + 1$. Is I a prime ideal?

Problem 5.
Show that two free groups are isomorphic if and only if they have equal ranks.

Problem 6.
Find the $\mathbb{Q}$-dimension of the splitting field over $\mathbb{Q}$ of $x^5 - 3$.