Fall 2012: PhD Analysis Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1:
Show that the space of all continuous functions on the interval $[0, 1]$ with the sup norm $||f|| = \max |f(x)|$ is not a Hilbert space.

Problem 2. Suppose φ is a real-valued continuous function on the interval $[0, 1]$, and T is a linear operator on $L^2 [0, 1]$ given by

$$(Tf)(x) = \varphi(x) \int_0^1 \varphi(t) f(t) \, dt$$

for all $f \in L^2 [0, 1]$. Show that

(a) T is self-adjoint.

(b) there exists a number $\lambda \geq 0$ such that $T^2 = \lambda T$.

(c) Find the spectral radius $r(T)$ of T.

Problem 3. Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$. Show that

(a) If $\|T\| \leq 1$, then T and its adjoint operator T^* have the same fixed point. i.e. Show that for $x \in \mathcal{H}$,

$$Tx = x \iff T^* x = x.$$

(b) Let λ be an eigenvalue of T. Is it true that its complex conjugate $\bar{\lambda}$ must be an eigenvalue of T^*? Is it true that $\bar{\lambda}$ must be in the spectrum of T^*? Justify your answers.

Problem 4. The heat kernel on $\mathbb{R}^3$ is given by $H_t(x) = (4\pi t)^{-3/2} e^{-|x|^2/(4t)}$ where $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^3$. Prove that if $u \in L^3(\mathbb{R}^3)$, then $t^{1/2}\|H_t * u\|_{L^\infty(\mathbb{R}^3)} \to 0$ as $t \to 0^+$. (Note that $*$ denotes convolution.)

Problem 5. Let M be a bounded subset of $C\,[a,b]$ with the sup norm and

$$A = \left\{ F\,(x) = \int_a^x f\,(t)\,dt : f \in M \right\}.$$

Show that A is a precompact subset of $C\,[a,b]$.

Problem 6. Consider the one-dimensional function f. Prove that if

$$\int_{-\infty}^{\infty} |\hat{f}(k)|^2 (1 + |k|^2)^s dk < \infty$$

for some $s > 3/2$ then f is globally Lipschitz, i.e., there exists a constant K such that $|f(x) - f(y)| \le K|x - y|$.