Spring 2014: PhD Algebra Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1: Find the smallest order of a group which is not cyclic and not isomorphic to a subgroup of S_5 (the symmetric group on five objects).

Problem 2: Consider the four dimensional real vector space

$$V = \{f : \mathbb{Z}/4\mathbb{Z} \to \mathbb{R}\}$$

with the $\mathbb{R}[x]$-module structure given by shifting so that $(xf)(r) = f(r+1)$. Find a direct sum decomposition of V into irreducible $\mathbb{R}[x]$-modules.

Problem 3: Let p, q be prime numbers with $p < q$ such that p is not a divisor of $q-1$. Let G be a group of order qp. Which of the following is true: (a) G is always simple. (b) G is never simple. (c) G could be simple or non-simple.

Problem 4: Give an example of a ring R and an R-module M which is projective but not free.

Problem 5: Let V be a linear representation of a group G over the field $\mathbb{Q}(\sqrt{2})$, and let $\chi : G \to \mathbb{Q}(\sqrt{2})$ be its character. Then V is also a vector space $V_{\mathbb{Q}}$ over $\mathbb{Q}$, and is again a linear representation of G. Express its character $\chi_{\mathbb{Q}}$ in terms of the original character χ.

Problem 6: Let $\mathbb{C}(x)$ be the field of complex rational functions, i.e., the fraction field of the polynomial ring $\mathbb{C}[x]$. Let $\mathbb{C}(y)$ be another copy of the same field in the variable y. This field is an algebra over $\mathbb{C}$, hence $\mathbb{C}(x) \otimes_{\mathbb{C}} \mathbb{C}(y)$ is another algebra over $\mathbb{C}$. Is it also a field?