

Math 108 Spring 2020 Final Exam
Due Wednesday June 10 at Midnight (48 hours take home)

1. (15 points) Determine the three pairs of equivalent sentences below and find their truth tables:

- (a) P
- (b) $P \wedge (\sim Q)$
- (c) $\sim ((\sim P) \wedge Q)$
- (d) $\sim [P \implies (P \wedge Q)]$
- (e) $P \vee [(\sim Q) \vee (P \vee (\sim Q))]$
- (f) $P \iff [(Q \implies P) \vee ((\sim Q) \implies P)]$

2. (15 points) (Induction) In the universe \mathbb{N} prove that

$$(\forall n) \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}.$$

3. (20 points) Consider three possible universes: \mathbb{N}, \mathbb{Z} and \mathbb{R} .

- (a) Determine for each of the following four sentences and each of the three above universes whether the sentence is true in the universe.

- i. $(\forall x)(\exists!y) x^3 = y^2$
- ii. $(\forall x)(\exists!y) x^2 = y^3$
- iii. $(\exists!y)(\forall x) xy^2 = y$
- iv. $(\exists!y)(\forall x) xy^2 = x$

- (b) Prove one case in which the sentence is true.
(c) Prove one case in which the sentence is false.

4. (15 points) Prove that if n is a natural number then n is a multiple of three iff $n^2 - 1$ is not a multiple of three.

5. (15 points) Consider the relation S from \mathbb{R} to \mathbb{R} given by xSy if $x - y \in \mathbb{Z}$.

- (a) Show that S is an equivalence relation.
(b) Find three different real numbers a, b and c for which $\bar{a} = \bar{b} \neq \bar{c}$.

6. (20 points) If A is a set consider the relation $R = \{((x, y), \{x, y\}) \mid (x \in A) \wedge (y \in A)\}$ from $A \times A$ to $\mathcal{P}A$.

- (a) Draw an arrow diagram (eg Fig 3.1.1.b) for R if $A = \{1, 2, 3\}$.
(b) Show that for any set A the relation R is a function.
(c) Show that for any set A the relation R is not onto.
(d) Show that for any set A with at least two elements the relation R is not one-to-one.