Preliminary Exam in Analysis

Spring, 2017

Instructions:

(1) All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.
(2) Use separate sheets for the solution of each problem.

Problem 1. Define $f_n : [0, \infty) \to \mathbb{R}$ by

$$f_n(x) = \sin \sqrt{x + 4n^2\pi^2},$$

and let $\mathcal{F} = \{f_n : n \in \mathbb{N}\} \subset C([0, \infty))$.

(a) Show that $\mathcal{F}$ is equicontinuous in $C([0, \infty))$.

(b) Show that the sequence (f_n) converges pointwise but not uniformly on $[0, \infty)$.

(c) Is $\mathcal{F}$ totally bounded in the space $C_b([0, \infty))$ of bounded, continuous functions $f : [0, \infty) \to \mathbb{R}$, equipped with the sup-norm?

Problem 2. Suppose that X is a metric space with metric d such that every continuous function $f : X \to \mathbb{R}$ is bounded. Prove that X is complete.

Problem 3. Let A be a linear operator on a Banach space B that maps any strongly converging sequence into a weakly converging one. Prove that A is a bounded operator.

Problem 4. Let $K(x, y)$ be a continuous function on the unit square. Prove that A is a compact operator acting on $L^2[0, 1]$, where

$$(Af)(x) = \int_0^1 K(x, y) f(y) dy.$$

Problem 5. Prove the Riemann-Lebesgue Lemma, namely if $f \in L^1(\mathbb{R}^n)$ then $\widehat{f}(\xi) \to 0$ as $|\xi| \to \infty$, where

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} f(x) dx.$$

Problem 6. Give an example of a sequence of functions that converges weakly in L^2, strongly in L^1, but does not converge strongly in L^2. Be sure to justify all of your assertions.