Fall 2011: MA Analysis Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1:
Let (X, d) be a metric space and let (x_n) be a sequence in X. For the purpose of this problem adopt the following definition: $x \in X$ is called a *cluster point* of (x_n) iff there exists a subsequence $(x_{n_k})_{k\geq 0}$ such that $\lim_k x_{n_k} = x$.

(a) Let $(a_n)_{n\geq 0}$ be a sequence of distinct points in X. Construct a sequence $(x_n)_{n\geq 0}$ in X such that for all $k = 0, 1, 2, \ldots$, a_k is a cluster point of (x_n).

(b) Can a sequence (x_n) in a metric space have an *uncountable* number of cluster points? Prove your answer. (If you answer yes, give an example with proof. If you answer no, prove that such a sequence cannot exists). You may use without proof that $\mathbb{Q}$ is countable and $\mathbb{R}$ is uncountable.

Problem 2:
Let X be a real Banach space and X^* its Banach space dual. For any bounded linear operator $T \in \mathcal{B}(X)$, and $\phi \in X^*$, define the functional $T^*\phi$ by

$$T^*\phi(x) = \phi(Tx), \text{ for all } x \in X.$$

(a) Prove that T^* is a bounded operator on X^* with $\|T^*\| \leq \|T\|$.

(b) Suppose $0 \neq \lambda \in \mathbb{R}$ is an eigenvalue of T. Prove that λ is also an eigenvalue of T^*. (Hint 1: first prove the result for $\lambda = 1$. Hint 2: For $\phi \in X^*$, consider the sequence of Cesàro means $\psi_N = N^{-1}\sum_{n=1}^{N} \phi_n$, of the sequence ϕ_n defined by $\phi_n(x) = \phi(T^n x)$.)

Problem 3:
Let $\mathcal{H}$ be a complex Hilbert space and denote by $\mathcal{B}(\mathcal{H})$ the Banach space of all bounded linear transformations (operators) of $\mathcal{H}$ considered with the operator norm.

(a) What does it mean for $A \in \mathcal{B}(\mathcal{H})$ to be *compact*? Give a definition of compactness of an operator A in terms of properties of the image of bounded sets, e.g., the set $\{Ax \mid x \in \mathcal{H}, \|x\| \leq 1\}$.

(b) Suppose $\mathcal{H}$ is separable and let $\{e_n\}_{n\geq 0}$ be an orthonormal basis of $\mathcal{H}$. For $n \geq 0$, let P_n denote the orthogonal projection onto the subspace spanned by $e_0, \ldots, e_n$. Prove that $A \in \mathcal{B}(\mathcal{H})$ is compact iff the sequence $(P_n A)_{n\geq 0}$ converges to A in norm.

Problem 4:
Let $\Omega \subset \mathbb{R}^n$ be open, bounded, and smooth. Suppose that $\{f_j\}_{j=1}^{\infty} \subset L^2(\Omega)$ and $f_j \rightharpoonup g_1$ weakly in $L^2(\Omega)$ and that $f_j(x) \to g_2(x)$ a.e. in Ω. Show that $g_1 = g_2$ a.e. (**Hint:** Use Egoroff's theorem which states that given our assumptions, for all $\epsilon > 0$, there exists $E \subset \Omega$ such that $\lambda(E) < \epsilon$ and $f_j \to g_2$ uniformly on E^c.)

Problem 5:
Let $u(x) = (1 + |\log x|)^{-1}$. Prove that $u \in W^{1,1}(0,1)$, $u(0) = 0$, but $\frac{u}{x} \notin L^1(0,1)$.

Problem 6:
Let $H = \{f \in L^2(0, 2\pi) \ : \ \int_0^{2\pi} f(x)dx = 0\}$. We define the operator Λ as follows:

$$(\Lambda f)(x) = \int_0^x f(y)dy\,.$$

(a) Prove that $\Lambda : H \to L^2(0, 2\pi)$ is continuous.

(b) Use the Fourier series to show that the following estimate holds:

$$\|\Lambda f\|_{H_0^1(0,2\pi)} \leq C\|f\|_{L^2(0,2\pi)}\,,$$

where C denotes a constant which depends only on the domain $(0, 2\pi)$. (Recall that $\|u\|^2_{H_0^1(0,2\pi)} = \int_0^{2\pi} |\frac{du}{dx}(x)|^2 dx$.)