Algebra Prelim Problems

Fall, 2014

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1.

Let G_1, G_2 be finite index subgroups of a group G. Show that the intersection $G_1 \cap G_2$ also has finite index in G.

Problem 2.

Let G be a finite group and $N \subseteq G$ be a subgroup of index p, where p is the smallest prime dividing $|G|$. Prove N is a normal subgroup of G ($N \triangleleft G$).

Problem 3.

Does the additive group $\mathbb{Q}$ admit an epimorphism to a nontrivial finite group? Justify your answer.

Problem 4.

List all ideals of $\mathbb{F}_p[x]/(x^2 + x - 6)$ when

(a) $p = 7$

(b) $p = 5$.

Problem 5.

Let ρ be a representation of a finite group G on a vector space V and let $v \in V$.

(a) Show that averaging $\rho_g(v)$ over G gives a vector $\overline{v} \in V$ which is fixed by G.

(b) What can you say about this vector when ρ is an irreducible representation?

Problem 6.

If R is a commutative ring with identity, and S a multiplicative subset, then every ideal J of $S^{-1}R$ is of the form $S^{-1}I$ for some ideal I of R. Is I uniquely determined by J? Why or why not?