Preliminary exam in Analysis
Fall 2 21

All problems are worth the same amount. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Important: Please use a di erent sheet for the solution to each problem.

1. If X is a Banach space and $T \in \mathcal{B}(X)$ is invertible (both T and $T^{\ 1}$ are bounded linear maps from X to X), show that there is $\varepsilon > 0$ such that if $S \in \mathcal{B}(X)$ and $\|S \quad T\|_{op} < \quad$ then S is also invertible. (The invertible operators in $\mathcal{B}(X)$ form an open subset).

2. Let $C\ (\mathbb{R})$ denote the space of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \to 0$ as $|x| \to \infty$, equipped with the sup-norm. A family of functions $\mathcal{F} \subset C\ (\mathbb{R})$ is said to be tight if for every $\quad > 0$ there exists $R > 0$ such that $|f(x)| < \quad$ for all $x \in \mathbb{R}$ with $|x| \geq R$ and all $f \in \mathcal{F}$. Prove that $\mathcal{F} \subset C\ (\mathbb{R})$ is precompact in $C\ (\mathbb{R})$ if it is bounded, equicontinuous, and tight.

3. Consider the operator A on the Hilbert space $\ell^2(\mathbb{N})$ defined by

$$A(x_1, x_2, x_3, x_4, ...) = (0, \frac{x_1}{1}, \frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, ...).$$

   Prove A is compact, has no nontrivial eigenvectors, and $\{0\} = \sigma(A)$.

4. Define a bounded linear operator A on $L^2([0, 1)]$ as follows:

$$Af(x) = \int^x f(y)dy.$$

   (a) Find the adjoint A^*

   (b) Find $\|A\|$.

   (c) Show that the spectral radius of A is equal to zero.

5. Suppose that $f \in L^1(\mathbb{R})$, $f > 0$, and define $\hat{f}(k) = \int_{\mathbb{R}} f(x)e^{\ 2\ ikx}dx$. Prove that $|\hat{f}(k)| < \hat{f}(0)$ for every $k \neq 0$.

6. Let $f : \mathbb{R} \to \quad$ be a Schwartz function. Show that $\ _{n\in\mathbb{Z}} f(2n) = \ _{n\in\mathbb{Z}} \hat{f}(n/2)$. Here $\hat{f}(k) = \int_{\mathbb{R}} f(x)e^{\ 2i\ kx}dx$.