1. Let $\mathbb{F}$ be a field. Suppose $G$ is a finite subgroup of the multiplicative group $\mathbb{F}\backslash\{0\}$, i.e. $G \subset \mathbb{F}^x$ and $|G| < \infty$. Prove $G$ is cyclic.

2. Give an example of a ring that is a unique factorization domain (UFD) but not a principal ideal domain (PID). Justify your answer by giving an ideal and showing it is not principal.

3. Let $R$ be a principal ideal domain (PID), let $M$ be an $R-$module such that the annihilator $ann(M) = (a)$. Suppose

$$a = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3},$$

for distinct primes $p_i \in R$, $\alpha_i \in \mathbb{Z}_{\geq 1}$. Let

$$M_i = \{m \in M | p_i^{\alpha_i} m = 0\}.$$

Prove

$$M = M_1 \oplus M_2 \oplus M_3.$$

4. a. Let $G$ be the subgroup of $\mathrm{GL}_3(\mathbb{Z}/3\mathbb{Z})$ – the group of invertible matrices with entries in $\mathbb{Z}/3\mathbb{Z}$ – of the form

$$G = \{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \text{ such that } a, b, c \in \mathbb{Z}/3\mathbb{Z}\}.$$

Find the order of this group and decide whether or not it is abelian.

b. Prove or disprove: a group of order 135 must be abelian.

5. Let $G$ be a finite group satisfying the following property: if $A, B$ are subgroups of $G$ then $AB$ is a subgroup of $G$. Prove that $G$ is a solvable group.

6. Suppose $K = F(\alpha)$ is a proper Galois extension of $F$ and assume there exists an element $\sigma \in \mathrm{Gal}(K/F)$ satisfying $\sigma(\alpha) = \alpha^{-1}$. Show that $[K : F]$ is even and that $[F(\alpha + \alpha^{-1}) : F] = [K : F]/2$.