Fall 2012: PhD Algebra Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1.
a) Can a vector space V over an infinite field be a finite union

$$V = \bigcup_{i=1}^{k} V_i,$$

where for each i, $V_i \neq V$?
b) Can the group $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ be a union of finitely many proper subgroups?

Problem 2. Let G be an abelian group with n generators. Show that every subgroup $H \subset G$ has a generating set consisting of at most n elements.

Problem 3. Let F be a field and let $P \subset F$ be the intersection of all subfields in F. Show that:
a) If F has characteristic 0 then $P \cong \mathbb{Q}$,
b) If F has characteristic $p > 1$ then $P \cong \mathbb{F}_p$.

Problem 4. Let R be a commutative ring and I an ideal in R. Prove or disprove: The set $\sqrt{I} = \{\, a \in R : \exists\, n \in \mathbb{N},\, n > 0,\, a^n \in I \,\}$ is an ideal.

Problem 5. Find the number of field homomorphisms $\phi\colon \mathbb{Q}(\sqrt[3]{2}) \to \mathbb{C}$.

Problem 6.
Consider the dihedral group $D_4 = \langle\, r, s : s^2 = r^4 = 1,\ rs = sr^{-1} \,\rangle$ of order 8.
a) Find the conjugacy classes of D_4.
b) Find the character table of D_4.