PRELIMINARY EXAMINATION FALL 2010

ALGEBRA

Problem 1. Let G be a group which admits a finite set of generators. Show that G is countable.

Problem 2. Let G be a finite group. Show that G embeds in $GL(n, \mathbb{Z})$ for some n.

Problem 3. Show that the (multiplicative) group of $n \times n$ upper-triangular matrices (with real entries), having diagonal elements that are non-zero, is solvable.

Problem 4. Consider the ring $R = \mathbb{Z}[x]$. Give an example, with a proof, of an ideal in R which is not principal and of an ideal that is not prime.

Problem 5. Let R be a ring with identity. Recall that $x \in R$ is called *nilpotent* if $x^n = 0$ for some n. Prove that if x is nilpotent, then $1 + x$ is invertible.

Problem 6. Let F be a nontrivial finite extension field of $\mathbb{R}$. Prove that F is isomorphic to $\mathbb{C}$. You may use the fundamental theorem of algebra.