Problem 1.

If n and m are integers write $d_0(n,m) = |n-m|$ and $d_1(n,m) = 2^{-r}$ if $|n-m| = 2^r t$ with t odd if $n \neq m$ and $d_2(n,n) = 0$.

1. Prove that the metric space $(\mathbb{Z}, d_0)$ is complete.
2. Prove that the metric space $(\mathbb{Z}, d_2)$ is not complete.
   (Hint: Consider the sequence $\{a_n = \frac{4^n-1}{3}\}$).

Problem 2:

Assume that $f \in C^1(\mathbb{R})$ (so the derivative f' is continuous).

1. Show that there is a sequence of polynomials p_n so that
   (a) $\{p_n\}$ converges pointwise to f on $[0,1]$ and
   (b) $\{p_n'\}$ converges pointwise to f' on $[0,1]$.
2. Show that there is a sequence of polynomials q_n so that
   (a) $\{q_n\}$ converges pointwise to f on $[0,1]$ and
   (b) $\{q_n'\}$ does not converge pointwise to f' on $[0,1]$.

Problem 3:

Consider the Banach space $X = (C[0,1], \|\cdot\|_\infty)$ of continuous functions with the supremum norm and the Banach space $F = (B(X), \|\cdot\|_{op})$ of bounded linear functions from X to X with the operator norm. For each of the following functions $f_i : X \to X$ determine whether f_i is in F (bounded and linear) and if so find $\|f_i\|_{op}$. In the equations $\phi(x)$ is any element of X.

1. $f_1(\phi(x)) = \phi(x^2)$,
2. $f_2(\phi(x)) = \phi^2(x)$,
3. $f_3(\phi(x)) = \phi(x) + x^2$,
4. $f_4(\phi(x)) = x^2\phi(x)$.

Problem 4

Let $f, g \in L^1(\mathbb{R}^d)$. Prove that if

$$\int_E f \leq \int_E g$$

holds for every measurable set $E \subset \mathbb{R}^d$, then $f \leq g$ almost everywhere.

Problem 5

Let $f(t,x)$ be a function on $[0,\infty) \times \mathbb{T}$ that is continuously differentiable in t, and twice continuously differentiable in x. Here $\mathbb{T}$ is a circle. Suppose that the function satisfies the equation

$$\frac{\partial}{\partial t} f(t,x) = \frac{\partial^2}{\partial^2 x} f(t,x), \quad t > 0, x \in \mathbb{T}.$$

Show that

1. $\lim_{t\to\infty} \frac{\partial}{\partial x} f(t,x) = 0$,
2. For any $t > 0$, $f(t,x)$ is a smooth function of x.

Problem 6

For a bounded operator A on a Hilbert space $\mathcal{H}$, the numerical range of A is a subset N of $\mathbb{C}$ given by

$$N := \{(x, Ax), x \in \mathcal{H}, \|x\| = 1\},$$

where $(\cdot,\cdot)$ denotes the scalar product on $\mathcal{H}$.

1. Show that if A is Hermitian then $N \subset \mathbb{R}$.
2. Show that if λ belongs to the point spectrum of A then $\lambda \in N$.
3. Show that if A is a compact operator, and $\mathcal{H}$ is infinite dimensional then $0 \in \bar{N}$, where $\bar{N}$ is the closure of N.
4. Give an example of A for which N is not a closed set.