PRELIMINARY EXAM IN ANALYSIS
FALL 2019

All problems are worth the same amount. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Important: Please use a different sheet for the solution to each problem.

1. Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_n\}_{n\geq 1}$, and suppose $A \in \mathcal{B}(\mathcal{H})$ is such that

$$\sum_{n=1}^{\infty} \|Ae_n\|^2 < \infty.$$

Prove that A is compact.

2. Prove the following limit in the space of tempered distributions $\mathcal{S}^*(\mathbb{R})$:

$$\lim_{\varepsilon\to 0^+} \frac{1}{x+i\varepsilon} = \text{p.v.}\frac{1}{x} - i\pi\delta(x).$$

(Hint: use an approximation of the identity.)

3. Let f in $L^1([0,1])$ and $0 < p < \infty$. Let $f_p(x) = px^{p-1}f(x^p)$. Prove or disprove with a counterexample that $\lim_{p\to 1}\int_0^1 |f_p(x) - f(x)|dx = 0$.

4. Suppose $f \in C^1([0,1])$ and $f(0) = f(1) = 0$ and $\|f\|_{L^2([0,1])} = 1$. Show that

$$\|f'(x)\|_{L^2([0,1])}\|xf(x)\|_{L^2([0,1])} \geq 1/2.$$

5. Suppose $f \in L^2(\mathbb{R})$. Suppose $\int_{\mathbb{R}} f(y)e^{-y^2}e^{2xy}dy = 0$ for all $x \in \mathbb{R}$. Show that $f \equiv 0$.

6. Is there an isometry of the Hilbert space $L^2([0,1])$ to itself which preserves the subspace of polynomials but does not preserve their degree?