Spring 2014: PhD Analysis Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1: Let (g_n) be a sequence of absolutely continuous functions on $[0, 1]$ with $|g_n(0)| \leq 1$. Suppose also that for each n, $|g_n'(x)| \leq 1$ for Lebesgue almost everywhere $x \in [0, 1]$. Show that there is a subsequence of (g_n) that converges uniformly to a Lipschitz function on $[0, 1]$.

Problem 2: Let T be a linear operator from a Banach space X to a Hilbert space H. Show that T is bounded if and only if $x_n \rightharpoonup x$ implies that $T(x_n) \rightharpoonup T(x)$ for every weakly convergent sequence (x_n) in X.

Problem 3: Let $f, f_k : E \to [0, +\infty)$ be non-negative Lebesgue integrable functions on a measurable set $E \subseteq \mathbb{R}^n$. If (f_k) converges to f pointwise almost everywhere and

$$\int_E f_k dx \to \int_E f dx,$$

show that

$$\int_E |f - f_k| dx \to 0.$$

Problem 4: Let P_1 and P_2 be a pair of orthogonal projections onto H_1 and H_2, respectively, where H_1 and H_2 are closed subspaces of a Hilbert space H. Prove that $P_1 P_2$ is an orthogonal projection if and only if P_1 and P_2 commute. In that case, prove that $P_1 P_2$ is the orthogonal projection onto $H_1 \cap H_2$.

Problem 5: Let H be a (separable) Hilbert space with orthonormal basis $\{f_k\}_{k=1}^{\infty}$. Prove that the operator defined by

$$T(f_k) = \frac{1}{k} f_{k+1}, \quad k \geq 1,$$

is compact but has no eigenvalues.

Problem 6: Let $H_1 = L^2([-\pi, \pi])$ be the Hilbert space of functions $F(e^{i\theta})$ on the unit circle with inner product

$$(F, G) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{i\theta}) \overline{G(e^{i\theta})} d\theta.$$

Let H_2 be the space $L^2(\mathbb{R})$. Using the mapping

$$x \to \frac{i - x}{i + x}$$

of $\mathbb{R}$ to the unit circle, show that:
a) The correspondence $U : F \to f$, with

$$f(x) = \frac{1}{\pi^{1/2}(i+x)} F\left(\frac{i-x}{i+x}\right)$$

gives a unitary mapping of H_1 to H_2.
b) As a result,

$$\left\{\pi^{-1/2}\left(\frac{i-x}{i+x}\right)^n \frac{1}{i+x}\right\}_{n=-\infty}^{\infty}$$

is an orthonormal basis of $L^2(\mathbb{R})$.