PRELIMINARY EXAM IN ANALYSIS
SPRING 2018

All problems are worth the same amount. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Important: Please use a different sheet for the solution to each problem.

1. Let $L^2([0,1])$ be the Hilbert space of complex-valued square integrable functions with inner product
$$\langle f,g\rangle = \int_0^1 \overline{f(x)}g(x)dx.$$
Does $L^2([0,1])$ have an orthonormal basis $\{u_n | n \geq 0\}$ such that each $u_n(x)$ is a polynomial of degree $2n$ using only even powers of x, i.e.,
$$u_n(x) = a_0 + a_1x^2 + a_2x^4 + \cdot + a_nx^{2n}?$$
Construct such a basis if it exists, or prove that it does not exist.

2. Let $\mathcal{H}$ be a Hilbert space and let $P,Q \in \mathcal{B}(\mathcal{H})$ be two orthogonal projections. Prove that $\ker PQ \subseteq \ker P + \ker Q$ always, and that $\ker PQ = \ker P + \ker Q$ when PQ is also an orthogonal projection.

3. Let X be a Banach space. Suppose that $f : (0,1) \to \mathcal{B}(X)$ is a differentiable function in the sense that the limit
$$f'(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}$$
exists in the norm topology on $\mathcal{B}(X)$ for each t. Prove that $e^{f(t)}$ is also differentiable in the same sense, and that its derivative satisfies the inequality
$$\left\| \frac{d}{dt} e^{f(t)} \right\| \leq \|f'(t)\| e^{\|f(t)\|}.$$

4. Let T be a bounded linear operator on a Hilbert space with an orthonormal basis of eigenvectors with eigenvalues $\Lambda = \{\lambda_n\}$. Show that the spectrum $\sigma(T)$ is exactly the closure of the set Λ.

5. Let a be an irrational real number and let $f \in L^2(\mathbb{T}) = L^2(\mathbb{R}/2\pi\mathbb{Z})$ be a square-integrable function on the circle such that $f(\theta) = f(\theta + \pi a)$ as elements of $L^2(\mathbb{T})$. Prove that f is essentially constant.

6. Given $t > 0$ and given $\vec{x} \in \mathbb{R}^3$, let
$$K_t(\vec{x}) = \frac{e^{-|\vec{x}|^2/t}}{t}.$$
Show that if $f \in L^3(\mathbb{R}^3)$, then the convolution $K_t * f$ lies in $L^\infty(\mathbb{R}^3)$, and that its norm is bounded by a constant independent of t.