Q1__________ **Last Initial** ____

Q2__________ **Section** ____

(C01 Li, C02 Smothers, C03 Kshirsagar, C04 Shu, C05 Ming, C06 Smothers)

Q3__________ **Student ID** ____

Q4__________ **FULL Name** ____

Q5__________

Q6__________

Σ __________

FINAL EXAMINATION III

21D §C01-06, 6:00-8:00 pm

Wednesday December 9, 2015

Declaration of honesty: I, the undersigned, do hereby swear to uphold the very highest standards of academic honesty, including, but not limited to, submitting work that is original, my own and unaided by notes, peeking at the person next to me whose answer is probably wrong anyway, books, calculators, mobile phones, blackberries, blueberries, boysenberries, raspberries, artificial intelligence or any other electronic device. Volcanic-emotional-support-pet rocks without tattoos permitted.

Signature ______________________ **Date** ________________

Q1 scratch/extra space (do not erase your scratch computations, they might earn partial credit):

Question 1

The electric field of a particle at the origin with charge q is given by

$$\frac{q}{4\pi}\frac{\vec{r}}{|\vec{r}|^3}\,.$$

The electric field $\vec{E}$ of two charged particles at two different positions is obtained by adding the electric field of the first particle to the electric field of the second particle. **Write down** the electric field of a pair of particles where the first particle has charge q and is at the origin and the second particle has charge Q and is at position $\vec{a}$. **Compute** the divergence of this electric field. **Draw** a picture showing the two particles and four closed surfaces S_q, S_Q, S_{qQ} and S, where S_q contains the charge q, S_Q contains the charge Q, S_{qQ} contains both charges and S contains neither charge. **Calculate** the flux integral of $\vec{E}$ for each of these four surfaces. *Well-organized and explained responses will receive more credit.*

Q2 scratch/extra space (do not erase your scratch computations, they might earn partial credit):

Question 2

Let $D \subset \mathbb{R}^3$ be a region in space. A function $f : D \to \mathbb{R}$ is said to be *harmonic* if it obeys

$$\Delta f = 0\,,$$

where $\Delta f = \vec{\nabla} \cdot \vec{\nabla} f$.

(i) Suppose f is harmonic throughout D and the boundary of D is given by a closed surface $S = \partial D$. **Show** that

$$\oint_S dA\, \hat{N} \cdot \vec{\nabla} f = 0\,.$$

(ii) **Show** that if f is harmonic throughout D, then

$$\oint_S dA\, \hat{N} \cdot (f\, \vec{\nabla} f) = \int_D dV\, |\vec{\nabla} f|^2\,.$$

Q3 scratch/extra space (do not erase your scratch computations, they might earn partial credit):

Question 3

Suppose you are given a vector that depends smoothly on two parameters, *i.e.* $\vec{r} = \vec{r}(\alpha, \beta)$. Moreover you know that $\vec{r}(0,0) = j + 2k$. Use calculus to **write down** an approximation for $\vec{r}(.01, .003)$. **Find** approximately the area of the parallelogram with corners labeled by the vectors $\vec{r}(0,0)$, $\vec{r}(0, .003)$, $\vec{r}(.01, 0)$ and $\vec{r}(.01, .003)$. **Apply** both your formulæ to the case

$$\vec{r} = \alpha^2\, i + (1 + \alpha + \beta)\, j + (\alpha\beta + \beta + 2)\, k\,.$$

Q4 scratch/extra space (do not erase your scratch computations, they might earn partial credit):

Question 4

You probably already know that the volume of one hemisphere of a unit radius sphere is $2\pi/3$. Lets try to derive this result using the divergence theorem. Let the origin be at the center of a unit sphere and then **compute** the surface integral of the vector field $\vec{V} = \vec{r}$ over one hemisphere of the sphere (you may assume that the surface area of a unit sphere is 4π). **Explain** how to use the divergence theorem to rewrite this surface integral as a volume integral. **Calculate** this volume integral and use this to **find** the volume of a unit radius hemisphere.

Q5 scratch/extra space (do not erase your scratch computations, they might earn partial credit):

Question 5

Let $\vec{V}$ be any smooth vector field. **Show** that

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{V}) = 0\,.$$

Q6 scratch/extra space (do not erase your scratch computations, they might earn partial credit):

Question 6

Consider a vector field $\vec{W}$ representing a *linear flow* where $\vec{W}$ points in the same direction and has the same length at every point in space. **Find** a vector field $\vec{V}$ such that $\vec{W}$ equals the curl of $\vec{V}$ (you may assume that your x-axis and $\vec{W}$ point in the same direction). **Draw** pictures representing $\vec{V}$ and $\vec{W}$. Also **draw** two closed curves C_1 and C_2 such that $\oint_{C_1} d\vec{r} \cdot \vec{V} = 0$ and $\oint_{C_2} d\vec{r} \cdot \vec{V} > 0$. Now **draw** two surfaces Σ and Σ' such that

$$\int_{\Sigma} dA\, \hat{N} \cdot \vec{W} = \int_{\Sigma'} dA\, \hat{N} \cdot \vec{W} = \oint_{C_2} d\vec{r} \cdot \vec{V}\,.$$

Finally **draw** a surface S for which $\int_S dA\, \hat{N} \cdot \vec{W} = 0$. **Indicate** on every surface in your pictures the direction of the unit normal vector $\hat{N}$, and on each closed curve the direction of integration.