Research in Ramsey Theory and Automatic Theorem Proving

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Joint work with William J. Wesley and (faculty mentor) Prof. De Loera

October 12, 2021

Introduction - Overview

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- Solving the problem in a different way, and through computers.
- In this project, we are concerned with linear homogeneous equations with 3 variables. For example, $ax + by = cz$, where $a, b, c \in \mathbb{Z}$.

Introduction - Definitions with example

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If we are concerned about equation $x + y = z$ where solutions are within the bounds $[1, 6]$, then we can see that $2 + 4 = 6$ is a monochromatic solution.

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Definition (Rado Number)

For any equation D, the Rado Number $R_r(D)$ is the smallest N such that any $r-$coloring $\chi : \{1, 2, \ldots, N\} \rightarrow \{1, 2, \ldots, r\}$ must induce a monochromatic solution to D.

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We can avoid monochromatic solutions to equation $x + y = z$ if we $3-$color $1 - 13$ in the following way:

$$\{2, 3, 7, 12\}\{5, 6, 8, 9\}\{1, 4, 10, 11, 13\}$$

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In fact, $3-$coloring Rado Number for $x + y = z$ is 14.

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- Let computers do the heavy lifting for us.

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- Let me introduce some basic Boolean algebra terminology.

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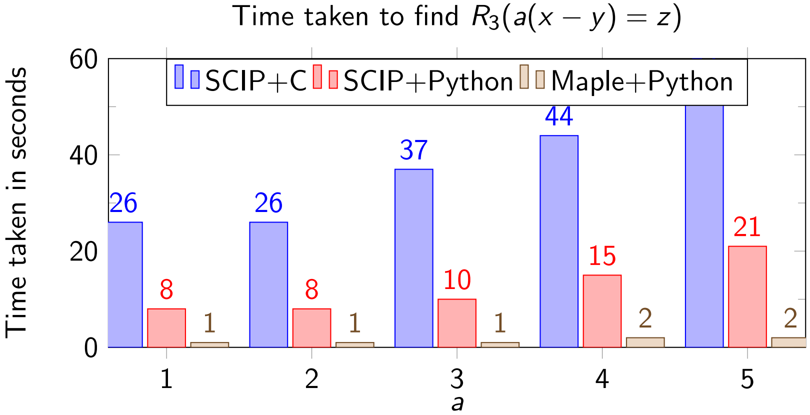
- Multi-threaded SAT solvers (Glucose).

Speedup results



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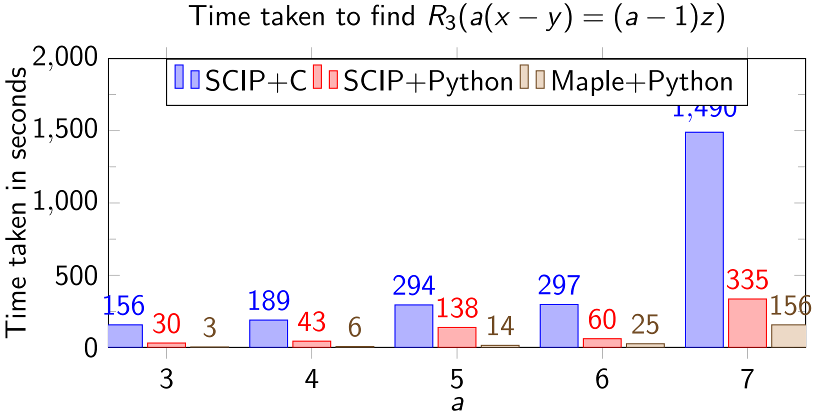

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- I am grateful for the financial support provided NSF grant 1818969 to Prof. Jesus De Loera. The National Science Foundation's Summer Scholars Internship Program provided me with this opportunity and funding for this project. Thanks to Professor Jesús De Loera for nominating me to this summer program.

Thank you!

Thank you very much everyone for your time!