MAT 150A: MODERN ALGEBRA

Final exam. December 12, 2018.

Name:____

SID: ____________________

Problem	1	2	3	4	5	6	7	8	9	10	Total
Points	/10	/10	/10	/10	/10	/10	/10	/10	/10	/10	/100

Read carefully the following instructions:

1. PLEASE DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO.
2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. PLEASE KEEP YOUR WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.
3. No notes, books or classmates may be used as resources for this exam.
4. Read directions to each problem carefully. **Show all your work for full credit.** In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of getting a good score in this exam. Neatness and organization are also important.
5. Make sure you have 7 pages, including this cover page.
6. **There are problems on both sides of the paper!!**
7. **You can use any statement that we have proved in class. If you do use a statement, make sure to explicitly state it. If you want to use a result that has not been proven in class, you will have to prove it. In particular, if you want to use a result that has appeared in your practice final, you'll have to prove it.**
8. There is plenty of space to work in this exam. You may NOT use scrap paper.
9. You have until 8pm sharp to finish this exam.
10. The exam is worth 100 points.

Good luck!

Problem 1 *(a) Can a group have class equation*

$$|G| = 1 + 4 + 5 + 5?$$

If your answer is yes, find such a group. If your answer is no, explain why. [4 points]

(b) The class equation of a group G is

$$|G| = 1 + 4 + 5 + 5 + 5$$

(b1) Show that G has an element of order 5*. [3 points]*

(b2) Does G have a normal subgroup with 4 *elements? [3 points]*

Problem 2 *The trace of a 2×2 matrix is defined to be*

$$\mathrm{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$$

(a) Show that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ *for any two 2×2 matrices A and B. [5 points]*

(b) Show that $\mathrm{tr}(XAX^{-1}) = \mathrm{tr}(A)$ *for any 2×2 matrix A and $X \in \mathrm{GL}_2(\mathbb{R})$. [5 points]*

Problem 3 *Consider the element*

$$p = (1423)(5276)(14)(15) \in S_7$$

(a) Express p as a product of disjoint cycles.[2 points]

(b) Find the number of elements in the conjugacy class of p. (It is ok to leave your answer in terms of factorials and combinations) [4 points]

(c) Is p even? If so, find the number of elements in the conjugacy class of p <u>*in* A_7.</u> *[4 points]*

Problem 4 *Consider the following matrices:*

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

(a) Which ones of the matrices above are orthogonal? [6 points]

(b) Which ones are special orthogonal? [4 points]

Problem 5 *Find all the solutions to the following system of congruences [10 points]*

$$\begin{aligned} x &\equiv 2 \mod 4 \\ x &\equiv 3 \mod 7 \\ 2x &\equiv 1 \mod 9 \end{aligned}$$

Problem 6 *(a) Can you find two non-isomorphic groups of order* 6*? If so, give an example, and explain why they are not isomorphic. If not, explain why. [5 points]*

(b) Can you find two non-isomorphic groups of order 7*? If so, give an example, and explain why they are not isomorphic. If not, explain why. [5 points]*

Problem 7 *Let G, H be groups with $|G| = 12$ and $|H| = 16$. Let $\varphi : G \to H$ be a homomorphism. Give a list of ordered pairs with all possible values of $(|\ker(\varphi)|, |\operatorname{image}(\varphi)|)$. Explain why your list is complete. [10 points]*

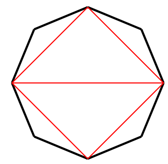
Problem 8 *Consider the set*

$$M := \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a^2 + b^2 \neq 0 \right\}$$

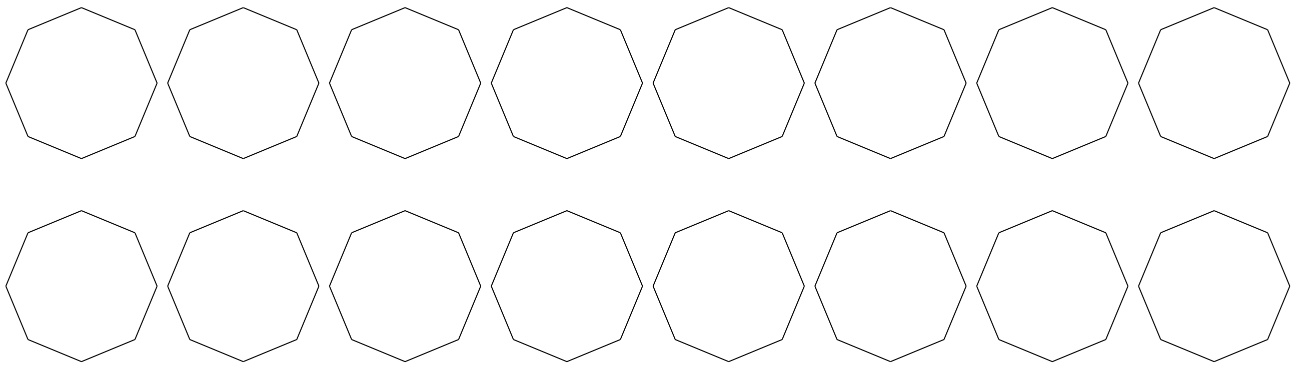
(a) Show that M is a subgroup of $\mathrm{GL}_2(\mathbb{R})$ *[5 points]*

(b) Show that $M \cap \mathrm{SL}_2(\mathbb{R}) = \mathrm{SO}_2$. *[5 points]*

Problem 9 *Consider the following triangulation T of a regular octagon.*



(a) Draw the triangulations in the orbit of T under the action of D_8 (note: probably you don't need to use all 16 octagons below) [5 points]



(b) Describe the stabilizer of the triangulation T, and verify the counting formula. [5 points]

Problem 10 *Let $n > 0$.*

(a) *Show that the set of all rotations in D_n is a normal subgroup. [5 points]*

(b) *Is the set of all reflections a normal subgroup of D_n? Justify your answer. [5points]*

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