Fall 2009: PhD Algebra Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1. Recall that an integral domain R is said to be a *unique factorization domain* if every element $x \in R$ can be written as a product of irreducible elements $\prod_{i=1}^{m} p_i$, and if the p_i are uniquely determined up to reordering and multiplication by units. Show that if R is a unique factorization domain, then every irreducible element generates a prime ideal.

Problem 2. The field extensions $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ and $\mathbb{Q}(\sqrt{\sqrt{2}})/\mathbb{Q}(\sqrt{2})$ are both Galois (you do not need to prove this). Show that $\mathbb{Q}(\sqrt{\sqrt{2}})/\mathbb{Q}$ is not Galois. For concreteness, assume the square roots are positive.

Problem 3. Let A and B be linear transformations on a finite dimensional vector space V. Prove that the dimension of kernel(AB) is less than or equal to the dimension of kernel(A) plus the dimension of kernel(B).

Problem 4. Let G be a group and H and K subgroups such that H has finite index in G. Prove that the intersection of K and H has finite index in K.

Problem 5. Prove that the algebra $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to the algebra $\mathbb{C} \oplus \mathbb{C}$.

Problem 6. If V is a finite-dimensional linear representation of a group G, then by definition the character function $\chi(g)$ is the trace of the action of g. This is usually studied when V is a complex vector space, but it is well-defined over any field. Find an example of a non-trivial representation V of a group G over some field F, such that $\chi(g) = 0$ for all g. (Non-trivial means that not all of the elements of G act by the identity.)