Spring 2010: PhD Analysis Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1:
Let (X,d) be a complete metric space, $\bar{x} \in X$ and $r > 0$. Set $D := \{x \in X : d(x, \bar{x}) \leq r\}$, and let $f : D \to X$ satisfying

$$d(f(x), f(y)) \leq k\, d(x, y)$$

for any $x, y \in D$, where $k \in (0, 1)$ is a constant.
Prove that if $d(\bar{x}, f(\bar{x})) \leq r(1 - k)$ then f admits a unique fixed point. (Guidelines: Assume the Banach fixed point theorem, also known as the contraction mapping theorem.)

Problem 2:
Give an example of two normed vector spaces, X and Y, and of a sequence of operators, $\{T_n\}_{n=0}^{\infty}$, $T_n \in L(X, Y)$ ($L(X, Y)$ is the space of the continuous operators from X to Y, with the topology induced by the operator norm) such that $\{T_n\}_{n=0}^{\infty}$ is a Cauchy sequence but it does not converge in $L(X, Y)$. (Notice that Y cannot be a Banach space otherwise $L(X, Y)$ is complete.)

Problem 3:
Let (a_n) be a sequence of positive numbers such that

$$\sum_{n=1}^{\infty} a_n^3$$

converges. Show that

$$\sum_{n=1}^{\infty} \frac{a_n}{n}$$

also converges.

Problem 4: Suppose that $h : [0,1]^2 \to [0,1]^2$ is a continuously differentiable function from the square to the square with a continuously differentiable inverse h^{-1}. Define an operator T on the Hilbert space $L^2([0,1]^2)$ by the formula $T(f) = f \circ h$. Prove that T is a well-defined bounded operator on this Hilbert space.

Problem 5: Let $H^s(\mathbb{R})$ denote the Sobolev space of order s on the real line $\mathbb{R}$, and let

$$\|u\|_s = \left(\int_{\mathbb{R}} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}}$$

denote the norm on $H^s(\mathbb{R})$, where $\hat{u}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} u(x) e^{-ix\xi} dx$ denotes the Fourier transform of u.

Suppose that $r < s < t$, all real, and $\epsilon > 0$ is given. Show that there exists a constant $C > 0$ such that

$$\|u\|_s \leq \epsilon \|u\|_t + C\|u\|_r \quad \forall u \in H^t(\mathbb{R}) .$$

Problem 6: Let $f : [0,1] \to \mathbb{R}$. Show that f is continuous if and only if the graph of f is compact in $\mathbb{R}^2$.