Fall 2006: PhD Algebra Preliminary Exam

Instructions:

(1) *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

(2) *Use separate sheets for the solution of each problem.*

Problem 1. Let G be a matrix group, and let $g \in G$ be an element with $\det(g) \neq 1$. Show that $g \notin G'$, the commutator group of G.

Problem 2. Let $A : V \to V$ be an operator on a finite-dimensional vector space V. Suppose A has characteristic polynomial $x^2(x-1)^4$ and minimal polynomial $x(x-1)^2$. What is the dimension of V? What are the possible Jordan forms of A?

Problem 3. Show that $\mathbb{Z}$ is a principal ideal domain.

Problem 4. Let G denote a finite abelian group. Let us consider the set G^* of all homomorphisms of the group G into the multiplicative group $\mathbb{C}^\times$ of nonzero complex numbers.

(a) Check that G^* can be considered as a group with respect to the operation of multiplication of homomorphisms.

(b) Prove that the group G^* is isomorphic to the group G.

Problem 5. Let us assign to every nonsingular complex 2×2 matrix A a transformation ϕ_A of the vector space Mat_2 of complex 2×2 matrices defined by the formula

$$\phi_A(X) = AXA^{-1}.$$

(a) Check that this formula specifies an action of the group $GL_2(\mathbb{C})$ of nonsingular complex matrices on Mat_2; moreover, it specifies a linear representation of this group.

(b) Prove that this representation is reducible.

(c) For every orbit of the above action, write down one element in that orbit, and find the corresponding stabilizer.

Problem 6. Consider the dihedral group D_9 (the group of isometries of regular 9-gons).

(a) Prove that D_9 cannot be represented as a direct product of two non-trivial groups.

(b) Determine if D_9 is solvable.

Fall 2006: PhD Analysis Preliminary Exam

Instructions:

(1) *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

(2) *Use separate sheets for the solution of each problem.*

Problem 1. Let $C([0,1])$ be the Banach space of continuous real-valued functions on $[0,1]$, with the norm $\|f\|_\infty = \sup_x |f(x)|$. Let $k : [0,1]\times[0,1] \to \mathbb{R}$ be a given continuous function. Let $T_k : C([0,1]) \to C([0,1])$ be the linear operator given by $T_k(f)(x) = \int_0^1 k(x,y)f(y)\,dy$.

(a) Show that T_k is a bounded operator.

(b) Find an expression for $\|T_k\|$ in terms of k.

(c) What is $\|T_k\|$ if $k(x,y) = x^2y^3$?

Problem 2. Let X be a metric space.

(a) Define X is sequentially compact.

(b) Define X is a complete metric space.

(c) Prove that a sequentially compact metric space X is complete.

(d) Let $B = \{\, x \,:\, \|x\|_2 \le 1 \,\}$ be the unit ball in $\ell^2(\mathbb{N})$. Show that B is not sequentially compact.

Problem 3. Give an example of a Banach space X and a sequence (x_n) of elements in X such that $\sum_{n=1}^{\infty} x_n$ converges unconditionally (converges regardless of order), but does not converge absolutely ($\sum_{n=1}^{\infty} |x_n|$ does not converge). Prove this.

Problem 4. Let $f \in L^2(\mathbb{T})$, and let $(\hat{f}_n)_{n\in\mathbb{Z}}$ be the Fourier coefficient sequence of f; here, $\mathbb{T} := \{\, z \in \mathbb{C} \,:\, |z| = 1 \,\}$. If $(\hat{f}_n) \in \ell^1(\mathbb{Z})$, does it follow that f is continuous? (In other words, is there a continuous function that is equivalent to f in $L^2(\mathbb{T})$?) Prove your assertion.

Problem 5. Find all solutions T of the equation $x^{2006}T = 0$ in the space of tempered distributions $S^*(\mathbb{R}^{\mathbf{1}})$.

Problem 6. In which of the following cases is the operator $A = i\frac{d}{dx}$ acting on $L^2([0,1])$ *symmetric*, *essentially self-adjoint*, *self-adjoint*? Justify your answers.

(a) $D_A = C^1[0,1]$
(the space of continuously differentiable complex-valued functions on $[0,1]$)

(b) $D_A = \{\, f \in C^1[0,1] \,:\, f(0) = f(1) \,\}$

(c) $D_A = \{\, f \in C^1[0,1] \,:\, f(0) = f(1) = 0 \,\}$