

Fall 2011: PhD Analysis Preliminary Exam

Instructions:

1. All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.
2. Use separate sheets for the solution of each problem.

Problem 1:

Let (X, d) be a metric space and let (x_n) be a sequence in X . For the purpose of this problem adopt the following definition: $x \in X$ is called a *cluster point* of (x_n) iff there exists a subsequence $(x_{n_k})_{k \geq 0}$ such that $\lim_k x_{n_k} = x$.

- (a) Let $(a_n)_{n \geq 0}$ be a sequence of distinct points in X . Construct a sequence $(x_n)_{n \geq 0}$ in X such that for all $k = 0, 1, 2, \dots$, a_k is a cluster point of (x_n) .
- (b) Can a sequence (x_n) in a metric space have an *uncountable* number of cluster points? Prove your answer. (If you answer yes, give an example with proof. If you answer no, prove that such a sequence cannot exist). You may use without proof that \mathbb{Q} is countable and \mathbb{R} is uncountable.

Problem 2:

Let X be a real Banach space and X^* its Banach space dual. For any bounded linear operator $T \in \mathcal{B}(X)$, and $\phi \in X^*$, define the functional $T^*\phi$ by

$$T^*\phi(x) = \phi(Tx), \text{ for all } x \in X.$$

- (a) Prove that T^* is a bounded operator on X^* with $\|T^*\| \leq \|T\|$.
- (b) Suppose $0 \neq \lambda \in \mathbb{R}$ is an eigenvalue of T . Prove that λ is also an eigenvalue of T^* . (Hint 1: first prove the result for $\lambda = 1$. Hint 2: For $\phi \in X^*$, consider the sequence of Cesàro means $\psi_N = N^{-1} \sum_{n=1}^N \phi_n$, of the sequence ϕ_n defined by $\phi_n(x) = \phi(T^n x)$.)

Problem 3:

Let \mathcal{H} be a complex Hilbert space and denote by $\mathcal{B}(\mathcal{H})$ the Banach space of all bounded linear transformations (operators) of \mathcal{H} considered with the operator norm.

(a) What does it mean for $A \in \mathcal{B}(\mathcal{H})$ to be *compact*? Give a definition of compactness of an operator A in terms of properties of the image of bounded sets, e.g., the set $\{Ax \mid x \in \mathcal{H}, \|x\| \leq 1\}$.

(b) Suppose \mathcal{H} is separable and let $\{e_n\}_{n \geq 0}$ be an orthonormal basis of \mathcal{H} . For $n \geq 0$, let P_n denote the orthogonal projection onto the subspace spanned by e_0, \dots, e_n . Prove that $A \in \mathcal{B}(\mathcal{H})$ is compact iff the sequence $(P_n A)_{n \geq 0}$ converges to A in norm.

Problem 4:

Let $\Omega \subset \mathbb{R}^n$ be open, bounded, and smooth. Suppose that $\{f_j\}_{j=1}^\infty \subset L^2(\Omega)$ and $f_j \rightharpoonup g_1$ weakly in $L^2(\Omega)$ and that $f_j(x) \rightarrow g_2(x)$ a.e. in Ω . Show that $g_1 = g_2$ a.e. (**Hint:** Use Egoroff's theorem which states that given our assumptions, for all $\epsilon > 0$, there exists $E \subset \Omega$ such that $\lambda(E) < \epsilon$ and $f_j \rightarrow g_2$ uniformly on E^c .)

Problem 5:

Let $u(x) = (1 + |\log x|)^{-1}$. Prove that $u \in W^{1,1}(0,1)$, $u(0) = 0$, but $\frac{u}{x} \notin L^1(0,1)$.

Problem 6:

Let $H = \{f \in L^2(0, 2\pi) : \int_0^{2\pi} f(x) dx = 0\}$. We define the operator Λ as follows:

$$(\Lambda f)(x) = \int_0^x f(y) dy.$$

(a) Prove that $\Lambda : H \rightarrow L^2(0, 2\pi)$ is continuous.

(b) Use the Fourier series to show that the following estimate holds:

$$\|\Lambda f\|_{H_0^1(0,2\pi)} \leq C \|f\|_{L^2(0,2\pi)},$$

where C denotes a constant which depends only on the domain $(0, 2\pi)$. (Recall that $\|u\|_{H_0^1(0,2\pi)}^2 = \int_0^{2\pi} \left| \frac{du}{dx}(x) \right|^2 dx$.)