

Spring 2011: MA Analysis Preliminary Exam

Instructions:

1. All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.
2. Use separate sheets for the solution of each problem.

Problem 1:

Let $\Omega = (0, 1)$, the open unit interval in \mathbb{R} , and consider the sequence of functions $f_n(x) = ne^{-nx}$. Prove that $f_n \not\rightarrow f$ weakly in $L^1(\Omega)$, i.e., the sequence f_n does not converge in the weak topology of $L^1(\Omega)$.

(Hint: Prove by contradiction.)

Problem 2:

Let $\Omega = (0, 1)$, and consider the linear operator $A = -\frac{d^2}{dx^2}$ acting on the Sobolev space of functions X where

$$X = \{u \in H^2(\Omega) \mid u(0) = 0, u(1) = 0\},$$

and where

$$H^2(\Omega) = \{u \in L^2(\Omega) \mid \frac{du}{dx} \in L^2(\Omega), \frac{d^2u}{dx^2} \in L^2(\Omega)\}.$$

Find all of the eigenfunctions of A belonging to the linear span of

$$\{\cos(\alpha x), \sin(\alpha x) \mid \alpha \in \mathbb{R}\},$$

as well as their corresponding eigenvalues.

Problem 3:

Let $\Omega = (0, 1)$, the open unit interval in \mathbb{R} , and set

$$v(x) = (1 + |\log x|)^{-1}.$$

Show that $v \in W^{1,1}(\Omega)$ and that $v(0) = 0$, but that $\frac{v}{x} \notin L^1(\Omega)$. (This shows the failure of Hardy's inequality in L^1 .) Note, that $W^{1,1}(\Omega) = \{u \in L^1(\Omega) \mid \frac{du}{dx} \in L^1(\Omega)\}$, where $\frac{du}{dx}$ denotes the weak derivative.

Problem 4:

Let $f(x)$ be a periodic continuous function on \mathbb{R} with period 2π . Show that

$$\hat{f}(\xi) = \sum_{n=-\infty}^{\infty} b_n \tau_n \delta \text{ in } \mathcal{D}', \quad (1)$$

that is, that equality in equation (??) holds in the sense of distributions, and relate b_n to the coefficients of the Fourier series. Note that δ denotes the Dirac distribution and τ_y is the translation operator, given by $\tau_y f(x) = f(x + y)$.

(**Hint:** Write $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ with convergence in $L^2(0, 2\pi)$ and where the coefficients $c_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-inx} f(x) dx$.)

Problem 5:

Let $f(x)$ be a periodic continuous function on \mathbb{R} with period 2π . Given $\epsilon > 0$, prove that for $N < \infty$ there is a finite Fourier series

$$\phi(x) = a_0 + \sum_{n=1}^N [a_n \cos(nx) + b_n \sin(nx)] \quad (2)$$

such that

$$|\phi(x) - f(x)| < \epsilon \quad \forall x \in \mathbb{R}.$$

This shows that the space of real-valued trigonometric polynomials on \mathbb{R} (functions which can be expressed as in (??)) are *uniformly* dense in the space of periodic continuous function on \mathbb{R} with period 2π .

(**Hint:** The Stone-Weierstrass theorem states that if X is compact in \mathbb{R}^d , $d \in \mathbb{N}$, then the algebra of all real-valued polynomials on X (with coordinates (x_1, x_2, \dots, x_d)) is dense in $C(X)$.)

Problem 6:

For $\alpha \in (0, 1]$, the space of Hölder continuous functions on the interval $[0, 1]$ is defined as

$$C^{0,\alpha}([0, 1]) = \{u \in C([0, 1]) : |u(x) - u(y)| \leq C|x - y|^\alpha, x, y \in [0, 1]\},$$

and is a Banach space when endowed with the norm

$$\|u\|_{C^{0,\alpha}([0,1])} = \sup_{x \in [0,1]} |u(x)| + \sup_{x,y \in [0,1]} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

Prove that the closed unit ball $\{u \in C^{0,\alpha}([0,1]) : \|u\|_{C^{0,\alpha}([0,1])} \leq 1\}$ is a compact set in $C([0,1])$.

(Hint: The Arzela-Ascoli theorem states that if a family of continuous functions U is equicontinuous and uniformly bounded on $[0,1]$, then each sequence u_n in U has a uniformly convergent subsequence. Recall that U is uniformly bounded on $[0,1]$ if there exists $M > 0$ such that $|u(x)| < M$ for all $x \in [0,1]$ and all $u \in U$. Further, recall that U is equicontinuous at $x \in [0,1]$ if given any $\epsilon > 0$, there exists $\delta > 0$ such that $|u(x) - u(y)| < \epsilon$ for all $|x - y| < \delta$ and every $u \in U$.)