Preliminary Exam in Algebra

Spring, 2017

Instructions:

(1) All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.
(2) Use separate sheets for the solution of each problem.

Problem 1. Let G be a finite group and $H \subseteq G$ a subgroup such that $[G : H] = p$ where p is the smallest prime dividing $|G|$. Show that H is a normal subgroup of G.

Problem 2. Let k be a field, and let $f \in k[x]$ be of degree $= n \geq 1$. Let K be the splitting field of f (over k, embedded in some fixed algebraic closure of k). Prove that $[K : k] \leq n!$.

Problem 3. Show that the free group of rank 2 is not solvable.

Problem 4. Give an example of a projective R-module that is not free for $R = \mathbb{R}[x]/(x^4 + x^2)$.

Problem 5. Let G be the nonabelian group of order 57.
(a) How many 1-dimensional characters does G have ?
(b) What are the dimensions (aka degrees) of the other irreducible characters of G ?

Problem 6.
Let $\mathbb{F}$ be a finite field.
(a) Show that $|\mathbb{F}| = p^r$ for some prime p.
(b) Show that the multiplicative group $\mathbb{F}\backslash\{0\}$ is a cyclic group.