MA Algebra Preliminary Exam for 2005-06

Instructions: *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

Problem 1. Let field E be a finite extension of a field F, and let R be a subring of E that contains F. Prove that R is a field.

Problem 2. Dihedral group D_n is defined as a group of rigid motions on the Euclidean plane $\mathbb{R}^2$ preserving a regular n-gon in $\mathbb{R}^2$. Check that there exists a system of generators of this group consisting of two elements. Describe all endomorphisms (i.e. homomorphisms $D_n \to D_n$) of the group D_n. Calculate the number of endomorphisms (as a function of n).

Problem 3. Describe all possible Jordan forms of an $n \times n$ matrix X obeying $X^n = 0$.

Problem 4. Show that $\mathbb{Q}$ (the additive group of rational numbers) is not finitely generated.

Problem 5. Determine all finitely generated abelian groups which have finite group of automorphisms.

Problem 6. Suppose that $H \subset G$ is a subgroup which is contained in every nontrivial subgroup of G. Show that H is contained in the center of G.