Winter 2009: PhD Algebra Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1:
(a) Find a complex matrix M with

$$M^2 = \begin{bmatrix} 1 & 3 & -3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{bmatrix}$$

(b) Note that $-M$ is another solution. How many such matrices are there?

Problem 2:
Show that there are at least two nonisomorphic, nonAbelian groups of each of the orders 24 and 30.

Problem 3:
Show that if a finite group has exactly three conjugacy classes (noting that the identity forms one of the three) then the group has at most six elements. (Hint: Consider showing that in a finite group the number of elements in any conjugacy class divides the number of elements in the group.)

Problem 4: Let P be some set of prime numbers in the usual integers. Find a commutative ring R containing the integers such that the primes (irreducible elements) in R are precisely the elements of P up to multiplication by units. (Hint: What are all primes in the ring $\mathbb{Z}[\frac{1}{2}] = \{\frac{a}{2^r} | a \in \mathbb{Z}, 0 \leq r \in \mathbb{Z}\}$?)

Problem 5: Let A be the group of rational numbers under addition, and let M be the group of positive rational numbers under multiplication. Determine all homomorphisms from A to M.

Problem 6: An element of the ring of n-adic integers $\mathbb{Z}_n$ is an infinite (to the left) string of base n digits written $\ldots a_3 a_2 a_1 a_0.$ with $0 \leq a_i \leq n-1$. Addition and multiplication are defined as usual for the integers written in base n, except that one must carry indefinitely. (Note for example that in $\mathbb{Z}_{10}$ the element $\ldots 99999.$ is -1.)

Prove that $\mathbb{Z}_{10}$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_5$.