MA Algebra Preliminary Exam for 2004-05

Instructions: *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

Problem 1. Let F be a field, n, m be positive integers and A be an $n \times n$ matrix with coefficients in F. Suppose that $A^m = 0$. Show that $A^n = 0$.

Problem 2. Prove that $f(x) = x^4 + x + 1$ is irreducible over $\mathbb{Q}$.

Problem 3. Prove that $\mathbb{Q}$ contains no proper subgroups of finite index.

Problem 4. Give definition of PID and examples (without proofs) of:

a. A commutative ring which is a PID.

b. A commutative ring which is not a PID.

Problem 5. Let p be a prime number and $G = \mathbb{Z}_p$ be the finite cyclic group of order p. Prove that the group of automorphisms of G is cyclic and compute its order.

Problem 6. Let S_n denote the group of permutations of n objects. Find four different subgroups of S_4 isomorphic to S_3 and nine isomorphic to S_2.

PhD Algebra Preliminary Exam for 2004-05

Instructions: *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

Problem 1. Let F be a field, n, m be positive integers and A be an $n \times n$ matrix with coefficients in F. Suppose that $A^m = 0$. Show that $A^n = 0$.

Problem 2. Prove that $f(x) = x^4 + x + 1$ is irreducible over $\mathbb{Q}$.

Problem 3. Prove that $\mathbb{Q}$ contains no proper subgroups of finite index.

Problem 4. Let F be a functor from the category of sets into the category of sets. Prove that if for some non-empty set X, the set $F(X)$ is empty, then the set $F(Y)$ is empty for every set Y.

Problem 5. Let p be a prime number and $G = \mathbb{Z}_p$ be the finite cyclic group of order p. Prove that the group of automorphisms of G is cyclic and compute its order.

Problem 6. Let S_n denote the group of permutations of n objects. Find four different subgroups of S_4 isomorphic to S_3 and nine isomorphic to S_2.

Analysis Preliminary Exam for 2004-05

Instructions: *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

Problem 1. Show that the mapping $T : \mathbb{R} \to \mathbb{R}$ defined by

$$T(x) = \pi/2 + x - \arctan(x)$$

has no fixed points in $\mathbb{R}$ and that

$$|T(x) - T(y)| < |x - y|, \quad \text{for all distinct} \quad x, y \in \mathbb{R}$$

Why does not this example contradict the contraction mapping theorem?

Problem 2. Prove that the vector space $C([a, b])$ is separable.

Here and below, $C([a, b])$ is the vector space of continuous functions $f : [a, b] \to \mathbb{R}$ with the supremum norm

Problem 3. Suppose that $f_n \in C([a, b])$ is a sequence of functions converging uniformly to a function f. Show that

$$\lim_{n \to \infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$$

Give a counterexample to show that the pointwise convergence of continuous functions f_n to a continuous function f does not imply convergence of the corresponding integrals.

Problem 4. Let $\ell_2(\mathbb{Z})$ denote the complex Hilbert space of sequences $x_n \in \mathbb{C}, n \in \mathbb{Z}$, such that

$$\sum_{n=-\infty}^{\infty} |x_n|^2 < \infty.$$

Define the *shift operator* $S : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$ by

$$S((x_n)) = (x_{n+1}).$$

Show that S has no eigenvalues.

Problem 5. Consider the initial value problem

$$u'(t) = |u(t)|^\alpha, u(0) = 0.$$

Show that the solution of this problem is unique if $\alpha > 1$ and is not unique if $0 \leq \alpha < 1$.

Problem 6. Let $\mathcal{H}$ be a Hilbert space, $\mathcal{H}_0$ a dense linear subspace of $\mathcal{H}$, (x_n) a sequence in $\mathcal{H}$, and $x \in \mathcal{H}$ such that (i) there exists $M > 0$ such that $\|x_n\| \leq M$ for all n, (ii) $\lim_n \langle x_n, y \rangle = \langle x, y \rangle$, for all $y \in \mathcal{H}_0$. Prove that (x_n) converges to x in the weak topology of $\mathcal{H}$.