Spring 2012: PhD Analysis Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1:
For $u \in L^1(0,\infty)$, consider the integral

$$v(x) = \int_0^\infty \frac{u(y)}{x+y} dy$$

defined for $x > 0$. Show that $v(x)$ is infinitely differentiable away from the origin. Prove that $v' \in L^1(\epsilon, \infty)$ for any $\epsilon > 0$. Explain what happens in the limit as $\epsilon \to 0$.

Problem 2. Let $X \subset L^2(0, 2\pi)$ be the set of all functions $u(x)$ such that

$$u(x) = \lim_{K\to\infty} \sum_{k=-K}^{K} a_k e^{ikx} \text{ in } L^2\text{-norm, with } |a_k| \leq (1+|k|)^{-1}.$$

Prove that X is compact in $L^2(0, 2\pi)$.

Problem 3. For $\epsilon > 0$, we set

$$\eta_\epsilon(x) = \frac{1}{\pi} \sin\left(\frac{\epsilon \pi x}{x^2+\epsilon^2}\right) \frac{\epsilon}{x^2+\epsilon^2},$$

and define the convolution for $u \in L^2(\mathbb{R})$:

$$\eta_\epsilon * u\,(x) = \int_{\mathbb{R}} \eta_\epsilon(x-y) u(y) dy\,.$$

For $\epsilon > 0$, prove that $\sqrt{\epsilon}(\eta_\epsilon * u)(x)$ is bounded as a function of x and ϵ, and that $\eta_\epsilon * u$ converges strongly in $L^2(\mathbb{R})$ as $\epsilon \to 0$. What is the limit?

Problem 4. Let $u_n : [0,1] \longrightarrow [0,\infty)$ denote a sequence of measurable functions satisfying

$$\sup_n \int_0^1 u_n(x) \log(2 + u_n(x))\, dx < \infty\,.$$

If $u_n(x) \longrightarrow u(x)$ almost everywhere, show that $u \in L^1(0,1)$ and that $u_n \longrightarrow u$ in L^1 strongly. (**Hint.** One possible strategy is Egoroff's Theorem.)

Problem 5. Let $u : [0,1] \longrightarrow \mathbb{R}$ be absolutely continuous, satisfy $u(0) = 0$, and

$$\int_0^1 |u'(x)|^2 dx < \infty\,.$$

Prove that

$$\lim_{x \to 0^+} \frac{u(x)}{x^{\frac{1}{2}}}$$

exists and determine the value of this limit.

Problem 6. Consider on $\mathbb{R}^2$ the distribution defined by the locally integrable function

$$E(x,t) = \begin{cases} \frac{1}{2} & \text{if } t - |x| > 0 \\ 0 & \text{if } t - |x| < 0 \end{cases}\,.$$

Compute the distributional derivative

$$\frac{\partial^2 E}{\partial t^2} - \frac{\partial^2 E}{\partial x^2}\,.$$