Spring 2012: PhD Algebra Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1. Let A be a real $n \times n$ upper triangular matrix so that A commutes with its transpose A^T. Show that A is diagonal.

Problem 2. Suppose that G is a group which contains no index 2 subgroups. Show that every index 3 subgroup in G is normal.

Problem 3. Let F be a field and $F^\times$ be the multiplicative group of nonzero elements of F. Show that every finite subgroup of $F^\times$ is cyclic.

Problem 4.
Prove that $\mathbb{R}[X]/(X^2-1)\mathbb{R}[X] \approx \mathbb{R} \oplus \mathbb{R}$, but $\mathbb{R}[X]/(X^2-1)^2\mathbb{R}[X] \not\approx \mathbb{R} \oplus \mathbb{R}$.

Problem 5. Show that 9 and $6+3\sqrt{-5}$ do not have a greatest common divisor in $\mathbb{Z}[\sqrt{-5}]$.

Problem 6. Let F be a field, X an indeterminate, and let $F[[X]]$ denote the ring of formal power series with coefficients in F, where multiplication is defined as it is for polynomials. Prove that an element $s = a_0 + a_1X + \cdots \in F[[X]]$ is a unit in $F[[X]]$ if and only if $a_0 \neq 0$. Show that every ideal of $F[[X]]$ is of the form $X^nF[[X]]$ for some $n \geq 0$.