Spring 2011: PhD Algebra Preliminary Exam

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*

2. *Use separate sheets for the solution of each problem.*

Problem 1:
Let N be an $m \times m$ square matrix of complex numbers. Prove that the following conditions are equivalent:
(a) $NN^* = N^*N$, i.e., N is normal.
(b) N can be expressed as $N = A + iB$, where A and B are self-adjoint matrices of order $m \times m$ satisfying $AB = BA$ (and $i = \sqrt{-1}$).
(c) N can be expressed as $N = R\Theta$, where R and Θ are matrices of order $m \times m$ satisfying $R\Theta = \Theta R$, Θ is unitary and R is self-adjoint.

Problem 2:
Prove that a finite group G is abelian if and only if all its irreducible representations are 1-dimensional.

Problem 3:
Let

$$SL(2,\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\},$$

and let $PSL(2,\mathbb{Z})$ be the quotient group

$$PSL(2,\mathbb{Z}) := SL(2,\mathbb{Z})/\{\pm I\},$$

where I is the 2×2 identity matrix. Prove that $PSL(2,\mathbb{Z})$ is generated by the cosets of the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Hint: Note that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-k} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a - kc & b - kd \\ c & d \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}.$$

Problem 4:
Consider the group $G = \mathbb{Q}/\mathbb{Z}$. Show that for every natural number n the group G contains exactly one cyclic subgroup of the order n.

Problem 5:
Let R be a finite ring. Show that there exist n, m with $n > m$, so that

$$x^n = x^m$$

for all $x \in R$.

Problem 6:
Let J denote the ideal in $\mathbb{Z}[x]$ generated by 5 and the polynomial $p(x) = x^3 + x^2 + 1$. Determine if J is a maximal ideal.