Analysis Prelim Problems
Fall, 2014

Instructions:

1. *All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
2. *Use separate sheets for the solution of each problem.*

Problem 1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable. Suppose $|f(x)| \le 1$ and $|f''(x)| \le 1$ for all $x \in \mathbb{R}$. Prove or disprove that $|f'(x)| \le 2$ for all $x \in \mathbb{R}$.

Problem 2. Define the bounded linear operator $K : L^2(0,1) \to L^2(0,1)$ by

$$(Kf)(x) = \int_0^1 xy\,(1-xy)\,f(y)\,dy.$$

Find the spectrum of K and classify it.

Problem 3. Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, and let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis of $\mathcal{H}$. Define a metric $d : B \times B \to \mathbb{R}$ on the closed unit ball B of $\mathcal{H}$ by

$$d(x,y) = \sum_{n=1}^{\infty} \frac{|\langle x-y, e_n \rangle|}{2^n}.$$

(a) Show that a sequence (x_k) in B converges to $x \in B$ with respect to the metric d if and only if it converges weakly to x in $\mathcal{H}$.
(b) Prove that (B,d) is a compact metric space.

Problem 4. Let $C_0(\mathbb{R})$ denote the Banach space of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \to 0$ as $|x| \to \infty$, equipped with the sup-norm.
(a) For $n \in \mathbb{N}$, define $f_n \in C_0(\mathbb{R})$ by

$$f_n(x) = \begin{cases} 1 & \text{if } |x| \le n \\ n/|x| & \text{if } |x| > n \end{cases}$$

Show that $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ is a bounded, equicontinuous subset of $C_0(\mathbb{R})$, but that the sequence (f_n) has no uniformly convergent subsequence. Why doesn't this example contradict the Arzelà-Ascoli theorem?

(b) A family of functions $\mathcal{F} \subset C_0(\mathbb{R})$ is said to be *tight* if for every $\epsilon > 0$ there exists $R > 0$ such that

$$|f(x)| < \epsilon \qquad \text{for all } x \in \mathbb{R} \text{ with } |x| \geq R \text{ and all } f \in \mathcal{F}.$$

Prove that $\mathcal{F} \subset C_0(\mathbb{R})$ is precompact in $C_0(\mathbb{R})$ if it is bounded, equicontinuous, and tight.

Problem 5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a smooth (C^∞) function with compact support. Prove that

$$\lim_{n\to\infty} \left\{ \sqrt{\frac{8}{\pi}} \int_0^\infty n \sin\left(n^2 x^2\right) f(x)\, dx \right\} = f(0).$$

Hint. You can use the fact that

$$\lim_{a\to\infty} \left\{ \int_0^a \sin\left(t^2\right)\, dt \right\} = \sqrt{\frac{\pi}{8}}.$$

Problem 6.

(a) By choosing a suitable even, periodic extension for f, calculate the Fourier cosine series for $f(x) = \sin x, \quad x \in [0, \pi]$.

(b) Deduce that

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = 1/2.$$