# PRELIMINARY EXAM IN ALGEBRA
# SPRING 2020

All problems are worth the same amount. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Important: Please use a different sheet for the solution to each problem.

**1.** Let $R$ be a commutative ring with identity and $\mathcal{M}$ be a maximal ideal of $R$.

(a) Show $R/\mathcal{M}$ is a field.

(b) Show $\mathcal{M}$ is a prime ideal.

(c) Give an example showing the converse to (b) does not hold.

**2.** (a) Let $D$ be a UFD and let $a,b \in D$ be relatively prime elements such that $ab = c^n$ for some $c \in D$. Prove that there exist units $u_1, u_2 \in D$ and elements $\bar{a}, \bar{b} \in D$ such that $a = u_1\bar{a}^n$ and $b = u_2\bar{b}^n$.

(b) Use (a) to show that the equation $y^2 + y = x^3$ has only two integral solutions.

**3.** Suppose $R$ is a commutative ring with identity and that

$$0 \to L \to M \to N \to 0$$

is an exact sequence of $R$-modules. Prove that $M$ is Noetherian if and only if both $L$ and $N$ are Noetherian.

**4.** Let $K = \mathbb{Q}(\sqrt{5}, \sqrt{-7})$, and let $L$ be the splitting field over $\mathbb{Q}$ of $f(x) = x^3 - 10$.

a) Determine the Galois groups of $K$ and $L$ over $\mathbb{Q}$.

b) Decide whether $K$ contains a root of $f$.

c) Determine the degree of the field $K \cap L$ over $\mathbb{Q}$.

**5.** Let $G = D_8$, the dihedral group of order 8, which is generated by two elements $a, b$ with relations $a^4 = 1, b^2 = 1, bab^{-1} = a^{-1}$. Let $H \subset G$ be the subgroup isomorphic to $\mathbb{Z}_4$ generated by $a$.

(a) Find a complex basis of the tensor product $V = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} U$, where $U$ is the module over $\mathbb{C}[H]$ which is ismorphic to $\mathbb{C}$ as a vector space, and where $a$ acts by multiplication by $i = \sqrt{-1}$. Prove that it is a basis.

(b) Describe $V$ as a $\mathbb{C}[G]$-module. In particular, describe the matrix of multiplication by $a, b \in G$ in terms of the basis from the previous part.

**6.** Let H be a normal subgroup of a group $G$, and let $P$ be a subgroup of $H$. Assume that every automorphism of $H$ is inner. Prove that $G = H \cdot N_G(P)$