PRELIMINARY EXAM IN ALGEBRA
FALL 2019

All problems are worth the same amount. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Important: Please use a different sheet for the solution to each problem.

1. Let $G = GL_3(\mathbb{F}_3)$ be the general linear group of 3×3 matrices over the finite field $\mathbb{F}_3$.

(a) Show that there is no element $x \in G$ with $x^{27} = 1$.

(b) How many subgroups of order 27 does G have?

2. Show that the quaternion group $Q_8 = \langle i, j, k : i^2 = j^2 = k^2 = ijk \rangle$ of order 8 has a two-dimensional irreducible representation $Q_8 \to GL_2(\mathbb{C})$.

3. Let $\zeta = \exp(2\pi i/5)$, a primitive fifth root of unity.

(a) Find the degree of the field extension $[\mathbb{Q}[\zeta] : \mathbb{Q}]$, and describe a basis of $\mathbb{Q}[\zeta]$ as a vector space over $\mathbb{Q}$.

(b) Show that $G = \mathrm{Aut}(\mathbb{Q}[\zeta]/\mathbb{Q})$ is isomorphic to $\mathbb{Z}_4$.

(c) Describe the action of G as matrices in the basis from the first part.

(d) Let $H \subset G$ be the order two subgroup $\{1, a^2\}$, where a is a generator of G. Describe the field F of elements of $\mathbb{Q}(\zeta)$ preserved by this subgroup. What's the degree $[F : \mathbb{Q}]$?

4. (a) If R is a commutative ring with unit, and M, N are R-modules, how does one describe the R-module structure of $M \otimes_R N$?

(b) Let M be the module over $R = \mathbb{Z}[x]$ whose total space is $\mathbb{Z}$, such that x acts by multiplication by -1. Compute the tensor product $M \otimes_R M$ as an R-module.

5. (a) Prove or disprove: if R is a finite, commutative ring with unit, then it is a product of fields.

(b) Show that $\mathbb{Z}[i]/(41) \cong \mathbb{F}_{41} \times \mathbb{F}_{41}$.

6. Let $S \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ be a subset whose complement S' satisfies $(i, j) \in S'$ implies $(i+1, j) \in S'$ and $(i, j+1) \in S'$. For any field k, define an $R = k[x, y]$-module M_S by (i) As a k-vector space, M_S has a basis given by $e_{i,j}$ for $(i, j) \in S$.
(ii) the variables act by $x \cdot e_{i,j} = e_{i+1,j}$, $y \cdot e_{i,j} = e_{i,j+1}$, where $e_{i,j}$ is defined to be zero if $i, j \notin S$.

Answer the following:
(a) Check that this does in fact define an R-module M_S.
(b) Describe an isomorphism $M_S \otimes_R M_T \cong M_{S \cap T}$ using the universal property of tensor products. Define its inverse map explicitly, and prove that they are inverses.