MAT21D

Final

Name:____________________

Student ID:____________________

Problem	Points	Earned
1	8	
2	7	
3	6	
4	12	
5	10	
6	10	
7	10	
8	10	
Total	73	

Problem 1. *Double integrals.*

(a) (2 points) Evaluate $\displaystyle\int_0^{\pi}\int_x^{\pi}\frac{\sin y}{y}\,dy\,dx.$

(b) (3 points) A thin plate covers the region R bounded by $y = x$ and $y^2 = x$ in the first quadrant. Assume the plate has constant density δ and mass $M = 1$. Find δ.

(c) (3 points) Change to polar coordinates but do not evaluate.

$$\int_0^2\int_{-\sqrt{1-(y-1)^2}}^{0} xy^2\,dx\,dy$$

Problem 2. *Rectangular, Cylindrical and Spherical coordinates.*

(a) Set-up, but do not evaluate, volume integrals for the following solids.

(i) (2 points) *Cylindrical coordinates.* D is the right circular cylinder whose base is the circle $r = 3\cos\theta$ in the xy-plane and whose top lies in the plane $z = 5 - x$.

(ii) (2 points) *Spherical coordinates.* D lies above the cone $\phi = 3\pi/4$ and between the spheres $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 + z^2 = 4$.

(b) (3 points) Verify that for any real number $a > 0$ the rectangular equation $az = \sqrt{x^2 + y^2}$ is equivalent to the spherical equation $\phi = \arctan(a)$.

Problem 3. *Curvature.* The curvature κ of a smooth curve C is defined as the magnitude of the derivative of the curve's unit tangent vector $\mathbf{T}$ with respect to arc length s.

(a) (2 points) For a smooth parametrized curve

$$C:\ \mathbf{r}(t), \quad a \le t \le b,$$

state the formula for calculating κ.

(b) (4 points) Compute the curvature function $\kappa = \kappa(t)$ for the curve

$$\mathbf{r}(t) = (e^t \cos t)\,\mathbf{i} + (e^t \sin t)\,\mathbf{j} + \mathbf{k}.$$

Problem 4. *Generalizations.*

(a) (1 point) The line integral $\int_C f(x, y, z)\, ds$ generalizes the real line integral $\int_a^b f(x)\, dx$. Consider a parametrized curve

$$C: \ \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle, \quad a \le t \le b.$$

Write the equation relating ds and dt.

(b) (3 points) Consider the surface integral $\iint_S G(x, y, z)\, d\sigma$ where

$$S: \ \mathbf{r}(u, v) = \langle f(u, v), g(u, v), h(u, v) \rangle, \quad a \le u \le b, \ c \le v \le d.$$

Write the surface integral as a parametric surface integral.

(c) (4 points) Green's Theorem relates certain line integrals to surface integrals. Use Green's Theorem to write flux and circulation as double integrals in **del notation**.

Flux across C =

Circulation around C =

(d) (4 points) The Divergence Theorem and Stokes' Theorem generalize Green's Theorem as:

Flux of curl =

Flux across S =

Problem 5. *The del operator* ∇. Let all partial derivatives of scalar function $f = f(x, y, z)$ be continuous.

(a) (2 point) Write ∇ as a vector.

(b) (2 points) Write the vector produced by applying the del operator to f.

(c) (2 points) Suppose f is a potential function of vector field $\mathbf{F}$. Then in terms of f

$$\mathbf{F} \;=$$

and

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} \;=$$

(d) (4 points) If $\mathbf{F}$ is as in (c), show

$$\text{div}\,\text{curl}\,\mathbf{F} \;=\; \left|\text{curl}\,\text{grad}\,f\right|.$$

Problem 6. (10 points) *Surface Area.* Find the surface area of the cone $z = 1 + \sqrt{x^2 + y^2},\ z \leq 3$.

Problem 7. (10 points) *Exact differential forms.* Consider the line integral

$$\int_A^B df$$

where df is the differential form of scalar function

$$f(x, y, z) = y^2 + xz^2 + 3.$$

Does the value of the line integral depend on the path from A to B? Justify your answer.

Problem 8. (10 points) *Zero circulation.* Show $\mathbf{F} = \langle x, y, z \rangle$ has zero circulation around the boundary of any smooth orientable surface in space.