Math 108
Final Exam

Printed Name ______________________________
(FIRST) (LAST)

Signature ______________________________

Please Show All Your Work.

No Calculators -- No Scratch Paper -- No Cell Phones

There are **8 pages** of problems. (The last page is for extra credit.)

You are expected to do your own work, and to adhere to the UCD Code of Academic Conduct.

Please write your proofs as clearly and completely as possible.

Be sure to use complete sentences, and use appropriate connective words where they are needed.

If you use a proof by contradiction or a proof of the contrapositive, be sure to state clearly that you are doing this.

Please indicate clearly if you continue work on the back of a page

Have a Good Winter Break!

① LET $A = \{x, y, z\}$. LIST THE ORDERED PAIRS IN A RELATION ON A WHICH IS REFLEXIVE, SYMMETRIC, BUT <u>NOT</u> TRANSITIVE.

7 PTS

② FILL IN THE BLANKS TO PROVIDE EQUIVALENT PROPOSITIONAL FORMS:

6 PTS

A) $p \Rightarrow q \equiv$ ____ $\vee\, q$

B) $\sim(p \vee q) \equiv (\sim p)$ ____ $(\sim q)$

C) $p \Rightarrow q \equiv (\sim q) \Rightarrow$ ____

D) $\sim(p \wedge q) \equiv p$ ____ $(\sim q)$

E) $\sim(p \Rightarrow q) \equiv p$ ____ $(\sim q)$

F) $p \Rightarrow (q \vee r) \equiv p \wedge$ ____ $\Rightarrow r$

③ MATCH EACH OF THE FOLLOWING SETS WITH THE LETTER CORRESPONDING TO ITS CARDINALITY:

9 PTS

____ A) $[4, 9]$

____ B) $(6, \infty)$

____ C) $\mathbb{Q} \cup \{\pi\}$

____ D) $\mathbb{R} - \{0\}$

____ E) $\mathbb{N} \times \mathbb{N}$

____ F) $\mathcal{P}(\mathbb{N})$

____ G) $\mathcal{P}(\mathbb{R})$

____ H) $\mathcal{P}(\mathcal{P}(\{a, b\}))$

____ I) $\mathbb{R} \times \mathbb{R}$

a) $\aleph_0$	d) 16
b) 8	e) 4
c) c	f) NONE OF THE ABOVE

④ LET $n \in \mathbb{Z}$. PROVE THAT IF n^3 IS EVEN, THEN n IS EVEN.

5 PTS

⑤ FIND THE MOST USEFUL DENIAL OF THE FOLLOWING STATEMENT (MOVING THE NEGATION AS FAR TO THE RIGHT AS POSSIBLE):

$$(\forall \varepsilon)\{\varepsilon > 0 \Rightarrow (\exists M)[M > 0 \wedge (\forall x)(x \geq M \Rightarrow |f(x) - L| < \varepsilon)]\}.$$

10 PTS

⑥ LET $h: A \to B$ AND $g: B \to C$.
PROVE THAT IF $g \circ h$ IS 1-1, THEN h IS 1-1.

8 PTS

⑦ LET $X, Y \in \mathbb{R}$. PROVE THAT IF $X \notin Q$ AND $Y \in Q$, THEN $2X + Y^2 \notin Q$.

10 PTS

⑧ DEFINE $f: \mathbb{R} \to \mathbb{R}$ BY $f(x) = \begin{cases} 7 - 2x, & \text{IF } x < 2 \\ 9 - x^2, & \text{IF } x \geq 2. \end{cases}$

A) SHOW THAT f IS <u>NOT</u> 1-1.

5 PTS

B) SHOW THAT f IS ONTO.

8 PTS

(9) PROVE THAT $2^n > n^2$ FOR EVERY INTEGER $n \geq 5$ USING INDUCTION.

14 PTS

(10) DEFINE $f: \mathbb{R} - \{6\} \to \mathbb{R} - \{4\}$ BY $f(x) = \dfrac{4x+1}{x-6}$.

A) SHOW THAT f IS 1-1.

6 PTS

B) SHOW THAT f IS ONTO.

9 PTS

(11) DEFINE A RELATION S ON $\mathbb{R}$ BY $x S y$ IFF $x - y = k\pi$ FOR SOME $k \in \mathbb{Z}$. PROVE THAT S IS AN EQUIVALENCE RELATION ON $\mathbb{R}$.

10 PTS

(12) PROVE THAT $\sqrt{5}$ IS IRRATIONAL. (YOU MAY ASSUME THAT IF p IS PRIME AND $p \mid ab$, THEN $p \mid a$ OR $p \mid b$ FOR $a, b \in \mathbb{Z}$.)

13 PTS

(13) SHOW THAT THE RELATION $f: Z_4 \to Z_6$ GIVEN BY $f([x]) = [3x+5]$ IS A WELL-DEFINED FUNCTION.

9 PTS

(14) DEFINE A BIJECTION $f: (0,1) \longrightarrow (-\infty, 4)$. (YOU DO NOT HAVE TO SHOW THAT f IS BIJECTIVE)

6 PTS

(15) IF $U_1, \ldots, U_n$ ARE OPEN SETS IN A METRIC SPACE, SHOW THAT THEIR INTERSECTION $V = U_1 \cap \cdots \cap U_n$ IS ALSO OPEN.

9 PTS

⑯ LET S BE THE SET OF ALL INFINITE SEQUENCES WHICH HAVE ONLY 0, 4, OR 9 AS THEIR TERMS. PROVE THAT S IS UNCOUNTABLE.

11 PTS

⑰ GIVE A FORMULA FOR A BIJECTION f BETWEEN EACH OF THE FOLLOWING SETS. (YOU DO NOT HAVE TO PROVE THAT f IS BIJECTIVE.)

A) $f: \mathbb{N} \to \mathbb{Z}$

5 PTS

B) $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

5 PTS

C) $f: (-1,1) \to \mathbb{R}$

5 PTS

(18) PROVE THAT EVERY INTEGER $n \geq 2$ HAS A PRIME FACTOR USING THE WELL-ORDERING PRINCIPLE.

13 PTS

(19) LET $f: S \to T$, AND LET $A \subseteq S$ AND $C \subseteq T$. PROVE OR DISPROVE THE FOLLOWING:

a) IF $A \subseteq f^{-1}(C)$, THEN $f(A) \subseteq C$.

13 PTS

b) IF $C \subseteq f(A)$, THEN $f^{-1}(C) \subseteq A$.

(20) Define a relation S on $\mathbb{R}$ by xSy iff $x \geq 0$ or $y \leq 3x$.
Show whether or not S is a) reflexive b) symmetric c) transitive.

16
PTS
(EXTRA
CREDIT)

(21) Use the Cantor - Schroeder - Bernstein Theorem to show that $\mathbb{R} - \mathbb{Q} \approx \mathbb{R}$.

13
PTS
(EXTRA
CREDIT)