Name: ______________________

Final

Wait! Do not turn this page until told to.

No books, notes, phones, or calculators.

Show all of your work.

Justify every statement that you make.

Each of the 4 problems is worth 30 points.

Any points above 100 are a bonus.

Good luck!

1	2	3	4	

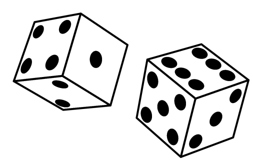
for office use

1. Let X, Y, Y' be discrete random variables.

   (a) Define the covariance of X and Y.

   (b) Prove that $\text{Cov}(X, Y + Y') = \text{Cov}(X, Y) + \text{Cov}(X, Y')$. Justify every step.

   (c) A die is rolled twice. Let Z equal the sum of the outcomes, and let W equal the first outcome minus the second. Are Z and W independent? Are Z and W uncorrelated?



2. A group of 400 people – 200 women and 200 men – is randomly divided into 2 groups of size 200 each.

   (a) What is the probability that all women will be in the same group, and all men in the other? Use the Stirling approximation to simplify your answer.

   (b) What is the probability that each group will have 100 women and 100 men? Use the Stirling approximation to simplify your answer.

3. Assume that raindrops are spherical, and let D be the diameter of a random raindrop. According to the *Marshall–Palmer Model*, the distribution of D is exponential with parameter λ.

   Recall that a ball of diameter D has volume $\frac{\pi}{6}D^3$ and surface area πD^2.

   (a) Prove that $E(D) = \frac{1}{\lambda}$.

   (b) We collected 1000 random raindrops, and their total volume is 3150 mm^3. We conclude that the average volume of a raindrop is 3.15 mm^3. Find the parameter λ.

   (c) Find the density function of the surface area of a random raindrop.

4. A set of 1000 cards numbered 1 through 1000 is randomly distributed among 100 people with each receiving 10 cards.

   (a) Are the following three events independent? (A) The card $\boxed{1}$ goes to the 2nd person; (B) the card $\boxed{3}$ goes to the 4th person; (C) the card $\boxed{5}$ goes to the 6th person.

   (b) Compute the expected value of the following random variable.
   X = the number of cards that are given to people whose age matches the number on the card.

   (c) Would you use a Poisson random variable to approximate the distribution of X? Explain.

Selected Distributions

Distribution	Probability Mass/Density	Support	Expectation
$\mathrm{B}(n,p)$	$p_X(k) \;=\; \binom{n}{k} p^k (1-p)^{n-k}$	$0, 1, \ldots, n$	np
$\mathrm{G}(p)$	$p_X(k) \;=\; (1-p)^{k-1} p$	$1, 2, 3, \ldots$	$\frac{1}{p}$
$\mathrm{Po}(\lambda)$	$p_X(k) \;=\; e^{-\lambda}\, \lambda^k / k!$	$0, 1, 2, \ldots$	λ
$\mathrm{U}(a,b)$	$f_X(t) \;=\; 1/(b-a)$	$t \in [a,b]$	$\frac{a+b}{2}$
$\mathrm{Exp}(\lambda)$	$f_X(t) \;=\; \lambda e^{-\lambda t}$	$t \in [0, \infty)$	$\frac{1}{\lambda}$
$\mathrm{N}(\mu, \sigma^2)$	$f_X(t) \;=\; \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(t-\mu)^2/2\sigma^2}$	$t \in \mathbb{R}$	μ

Useful Limits

$$\left(1 + \frac{x}{m}\right)^m \;\xrightarrow[m\to\infty]{}\; e^x$$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \;=\; e^x$$

$$n! \;\sim\; \frac{n^n}{e^n}\sqrt{2\pi n}$$