University of California Davis **Name (Print):** ____________________

Combinatorics MAT 145 **Student ID (Print):** ____________________

Final Examination **March 22 2019**

Time Limit: 120 Minutes

This examination document contains 9 non-blank pages, including this cover page, a total of 5 problems and two blank pages at the end, to be used for answers, if needed. You must verify whether there any pages missing, in which case you should let the instructor know.

Fill in all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam. You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) Let us consider a 11×19 rectangular grid as depicted in Figure 1, with marked points A, B in the corners and P, Q, R in the interior part of the grid. By definition, a staircase walk is a path in the grid which moves only *right* or *up*.

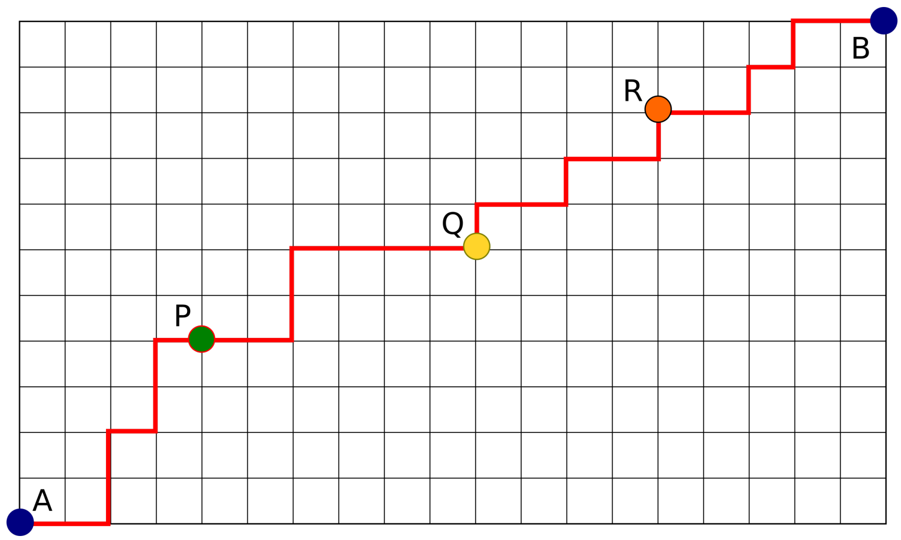

Figure 1: The 11×19 grid and a staircase walk from A to B passing through P, Q and R.

(a) (5 points) Find the number of staircase walks from A to B **avoiding** the point P.

(b) (5 points) Compute the probability that a staircase walk from A to B passes through all the three points P, Q **and** R.

(c) (5 points) How many staircase walks from A to B are there which either pass through the point P, **or** pass through the point Q **or** through the point R ?

(d) (5 points) Show that the number of paths from Q to R plus the number of paths from R to B equals the binomial coefficient $\binom{8}{3}$.

2. (20 points) Consider the graph $G = (V, E)$ in Figure 2.

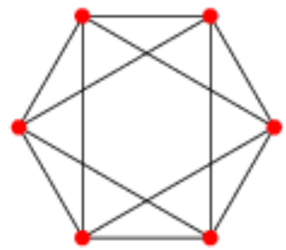

Figure 2: The graph for Problem 2.

(a) (4 points) Show that the graph G is planar.

(b) (4 points) Prove that there exists an Euler cycle in G.

(c) (4 points) Does there exist a Hamiltonian cycle in G ?
(Prove your answer.)

(d) (4 points) Prove that $\chi(G) = 3$, i.e. G is *not* bipartite *and* admits a 3-coloring.

(e) (4 points) Let T be a spanning tree for G. How many edges does T have ? (Prove your answer.)

3. (20 points) Solve the following three problems on labeled trees.

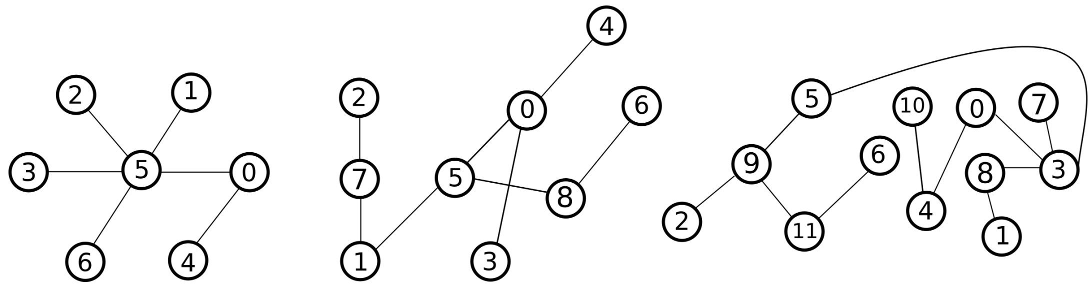

Figure 3: The trees T_1, T_2, T_3 for Problem 3.(a).

(a) (8 points) Find the Prüfer Code associated to the three trees depicted in Figure 3. The trees are denoted by T_1, T_2 and T_3 from left to right.

(b) (8 points) Draw the labeled trees associated to the following three Prüfer codes:

$$\{2, 1, 4\}, \quad \{2, 0, 3, 5, 6\}, \quad \{6, 7, 5, 1, 2, 4\}.$$

(c) (4 points) How many labeled trees are there with 7 nodes ?

4. (20 points) Solve the following three parts.

   (a) (10 points) Show that the chromatic polynomial $\pi_T(x)$ of a tree $T = (V, E)$ is

$$\pi_T(x) = x(x-1)^{|V|-1}.$$

   (b) (5 points) Find the number of 29-colorings of the graph in Figure 4.

Figure 4: The graph for Problem 4.(c).

   (c) (5 points) Find the number of spanning trees of the graph in Figure 5.

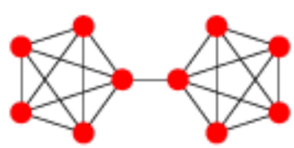

Figure 5: The graph for Problem 4.(c).

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5. (20 points) Decide whether each of the following statement is *true* or *false*. In the case the statement is *true*, provide an explanation, and in the case the statement is *false* provide a counter-example or an argument showing that the statement is false.

   (a) (4 points) The graph in Figure 6 is bipartite.

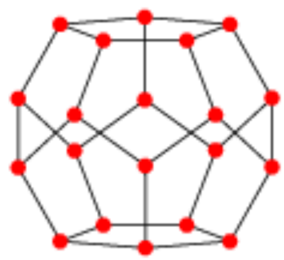

Figure 6: The graph for Problem 5.(a).

   (b) (4 points) Every bipartite graph $G = (V, E)$ such that each vertex $v \in V$ is of even degree, must have a perfect matching.

(c) (4 points) Let $k \in \mathbb{N}$ be a natural number and $k \geq 2$. Every finite bipartite graph $G = (V, E)$ such that every vertex $v \in V$ has degree k admits a perfect matching.

(d) (4 points) The graph G in Figure 7 is *not* planar.

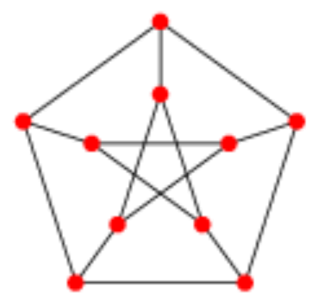

Figure 7: The graph for Problem 5.(c).

(e) (4 points) The polynomial $P(x) = x(x-1)(x-2)(x-3)$ cannot be the chromatic polynomial $\pi_G(x)$ of a planar graph G.