1. Let A be a bounded linear operator on a Hilbert space $\mathcal{H}$, and let $p(z)$ be a polynomial of complex coefficients on $\mathbb{C}$. Show that the spectrum

$$\sigma(p(A)) = p(\sigma(A))$$

where $p(\sigma(A)) := \{p(z) \, | z \in \sigma(A)\}$.

2. Let X be a normed linear space and Y be a Banach space. Let $K(X, Y)$ be the set of all compact operators from X to Y. Show that $K(X, Y)$ is a closed subspace of the space $\mathcal{B}(X, Y)$ (the space of all bounded linear operators from X to Y with the uniform topology) . You may assume that $K(X, Y)$ is a subspace of $\mathcal{B}(X, Y)$.

3. Let (X, Σ, μ) be a σ-finite measure space. Suppose f is a non-negative $\mu-$integrable function. For each $k \in \mathbb{N}$, let

$$E_k := \{x \in X | f(x) \geq k\}.$$

Show that

$$\sum_{k=1}^{\infty} \mu(E_k) < \infty.$$

4. Let $X = C([0, 1])$ with the norm $|| \cdot ||_{\infty}$. For any $f \in C([0, 1])$, let

$$\Phi(f) = \int_0^{1/2} f(x)dx - \int_{1/2}^1 f(x)dx.$$

Show that Φ is a bounded linear functional on X with $||\Phi|| = 1$. Moreover, show that Φ does not attain its norm in X. (i.e., there is no $f \in X$ with $||f||_{\infty} = 1$ such that $|\Phi(f)| = ||\Phi||$).

5. Let $f \in \mathcal{S}(\mathbb{R})$ be a Schwartz function on $\mathbb{R}$, and define

$$\|f\|_{BV} = \int_{\mathbb{R}} |f'(x)|dx.$$

Let $\hat{f}(k) = \int_{\mathbb{R}} e^{-2i\pi kx} f(x)dx$, for k a non-zero real number.

Show that

$$|\hat{f}(k)| \leq \frac{C}{|k|} \|f\|_{BV},$$

where C is some universal constant.

6. Suppose $K \in L^1(\mathbb{R})$, $\int_{\mathbb{R}} K(x)dx = 1$, and $\lim_{r \to 0} \int_{|x|>\delta} |K_r(x)|dx = 0$ for all $\delta > 0$, where $K_r(x) = \frac{1}{r} K(x/r)$. Show that for $1 \leq p < \infty$ and any $f \in L^p(\mathbb{R})$, we have

$$\lim_{r \to 0} \left\| \int_{\mathbb{R}} K_r(\cdot - y) f(y)dy - f(\cdot) \right\|_{L^p(\mathbb{R})} = 0.$$