1. Let (X, d) be a metric space, and let $\{f_n : X \to \mathbb{R}\}$ be an equicontinuous sequence of functions. Consider the set

$$C = \{x \in X \mid f_n(x) \text{ converges}\}.$$

   a) Prove C is closed.

   b) Let $X = [-1, 1]$ with the Euclidean metric. Give an example where $C = \{0\}$.

2. Let X be a separable Banach space. Show that there is an isometric embedding from X to $(\ell^\infty, ||\cdot||_\infty)$ (i.e. the space of bounded sequences).

3. Let (X, Σ, μ) be a measure space, and $a_k : X \to [0, \infty]$ be a $\mu-$measuable function for each k with

$$\int_X a_k(x) d\mu \leq \frac{1}{2^k}.$$

   Show that the series $\sum_{k=1}^\infty a_k(x)$ is convergent for $\mu-$almost every $x \in X$.

4. Consider the Hilbert space $L^2(\mathbb{T})$ of periodic L^2 functions on $[-\pi, \pi]$. For each of the following three sequences, determine and prove whether they converge strongly, weakly but not strongly, or diverge.

$$f_n(x) = n \cdot \chi_{[0,\frac{1}{n})} \qquad g_n(x) = n^{\frac{1}{2}} \cdot \chi_{[0,\frac{1}{n})} \qquad h_n(x) = e^{inx}.$$

5. Suppose f is a bounded monotonic function on $[-\pi, \pi]$. Show that $f \in H^s(\mathbb{T})$ for any $0 \leq s < \frac{1}{2}$.

6. Suppose $f \in C_0^\infty(\mathbb{R})$ such that $\left|\frac{d^n f}{dx^n}(x)\right| \leq 1$ for all $x \in \mathbb{R}$ for all integers $n \geq 1$ and $f(x) = 0$ for all $|x| > 1$. Show that $\hat{f}$ (the Fourier transform $\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} dx$) satisfies the following estimate

$$\left|\hat{f}(y)\right| \leq \frac{2}{(2\pi)^n} |y|^{-n} \text{ for all } y \in \mathbb{R} \text{ and all } n \geq 1.$$