1. Let k be a field, and let $B_n \subset GL_n(k)$ be the subgroup of upper triangular matrices with ones on the diagonal. Prove that B_n can be expressed as a semidirect product

$$B_n \cong H_1 \ltimes (H_2 \ltimes \cdots (H_{l-1} \ltimes H_l) \cdots),$$

where each group H_i is isomorphic to a vector space over k, with the group law being addition.

2. Let D be a Principal Ideal Domain (PID). Show that every proper ideal in D is a product $\mathfrak{m}_1 \mathfrak{m}_2 \cdots \mathfrak{m}_k$ of maximal ideals, which are uniquely determined up to order.

3. Let R be a commutative ring with unit, and let M, N be $R-$modules whose underlying set is finite. Prove that $M \otimes_R N$ is also a finite set.

4. (a) Let G be a finite group, and let H be a normal subgroup of G. Let P be a p-Sylow subgroup of H, and let K be the normalizer of P in G. Establish the equality $G = HK$.

   (b) Let G be a group of order 120, and let $H \subset G$ be a subgroup of order 24. Suppose that there is at least one left coset of H in G (other than H itself) that is also a right coset of H in G. Prove that H is a normal subgroup of G.

5. Let F be a field of characteristic p that is not perfect (so, not every element in F is a p-th power of another). Show that there exist inseparable irreducible polynomials in $F[X]$.

6. Suppose that E is a Galois extension of a field F such that $[E : F] = 5^3 \cdot 43^2$. Prove that there exist fields K_1 and K_2 lying strictly between F and E with the following properties:

   (a) each K_i is a Galois extension of F;

   (b) $K_1 \cap K_2 = F$;

   (c) $K_1 K_2 = E$.