Preliminary exam in Analysis
Fall 2021

All problems are worth the same amount. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Important: Please use a different sheet for the solution to each problem.

1. If X is a Banach space and $T \in \mathcal{B}(X)$ is invertible (both T and T^{-1} are bounded linear maps from X to X), show that there is $\varepsilon > 0$ such that if $S \in \mathcal{B}(X)$ and $\|S - T\|_{op} < \epsilon$ then S is also invertible. (The invertible operators in $\mathcal{B}(X)$ form an open subset).

2. Let $C_0(\mathbb{R})$ denote the space of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \to 0$ as $|x| \to \infty$, equipped with the sup-norm. A family of functions $\mathcal{F} \subset C_0(\mathbb{R})$ is said to be tight if for every $\epsilon > 0$ there exists $R > 0$ such that $|f(x)| < \epsilon$ for all $x \in \mathbb{R}$ with $|x| \geq R$ and all $f \in \mathcal{F}$. Prove that $\mathcal{F} \subset C_0(\mathbb{R})$ is precompact in $C_0(\mathbb{R})$ if it is bounded, equicontinuous, and tight.

3. Consider the operator A on the Hilbert space $\ell^2(\mathbb{N})$ defined by
$$A(x_1, x_2, x_3, x_4, ...) = (0, \frac{x_1}{1}, \frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, ...).$$
Prove A is compact, has no nontrivial eigenvectors, and $\{0\} = \sigma(A)$.

4. Define a bounded linear operator A on $L^2([0,1])$ as follows:
$$Af(x) = \int_0^x f(y)dy.$$
   (a) Find the adjoint A^*
   (b) Find $\|A\|$.
   (c) Show that the spectral radius of A is equal to zero.

5. Suppose that $f \in L^1(\mathbb{R})$, $f > 0$, and define $\hat{f}(k) = \int_{\mathbb{R}} f(x)e^{-2\pi ikx}dx$. Prove that $|\hat{f}(k)| < \hat{f}(0)$ for every $k \neq 0$.

6. Let $f : \mathbb{R} \to \mathbb{C}$ be a Schwartz function. Show that $\sum_{n\in\mathbb{Z}} f(2n) = \sum_{n\in\mathbb{Z}} \hat{f}(n/2)$. Here $\hat{f}(k) = \int_{\mathbb{R}} f(x)e^{-2i\pi kx}dx$.