Final Exam: Math 17B, CRN 53761, Monday, Dec 5 Fall 2022

Name:

Student ID:

You may use one page of notes.
You may not use a calculator.
You may not use the textbook.
Please do not simplify answers.

1. (25 points) Evaluate the following integrals:

   (a) $\int xe^{x^2-1}dx$

   (b) $\int_0^1 xe^{x-1}dx$

   (c) $\int_0^\infty e^{1-x}dx$

2. (50 points) The concentration of sulfuric acid in a lake T weeks after monitoring begins is $C(T)$ ppm with $C_0 = 100$ ppm. It follows a survival renewal equation: $C(T) = S(T)C_0 + \int_{t=0}^{T} S(T-t)R(t)dt$. The sulfuric acid has a survival rate of $S(\tau) = \frac{1}{(\mathbf{a}+\tau)^2}$ after τ weeks which depends on a parameter $\mathbf{a}$. A chemical plant is now raining sulfuric acid at a constant rate $R(t) = 10$ ppm per week.

   (a) Find $C(9)$ if $\mathbf{a} = 1$.

   (b) Find
   $$\lim_{T\to\infty} C(T).$$
   Your answer should depend on the parameter $\mathbf{a}$.

3. (25 points) Consider the differential equation

$$\frac{dy}{dx} = (xy^2)^{\frac{1}{3}}.$$

(a) Separate the differential equation to get an integral involving only y equal to one involving only x.

(b) Assume that $y = 0$ when $x = 0$ and find y when $x = 4$.

4. (50 points) Consider the Lotka-Volterra type system of differential equations with $F(t)$ the population of F as a function of time and $G(t)$ the population of G as a function of time. [This system might arise if the two compete for some resource but once the G population is high enough they are able to kill and eat F]:

$$\frac{dF}{dt} = F(3000 - G)$$

$$\frac{dG}{dt} = (F - 2000)(G - 1000)$$

(a) If $F = 0$ find the equilibrium value for G and determine whether it is stable under small changes to G.

(b) Sketch and label the lines in the G-F plane along which the direction field for the associated nonautonomous differential equation

$$\frac{dF}{dG} = \frac{F(3000 - G)}{(G - 1000)(F - 2000)}$$

is vertical and the lines along which it is horizontal (the nullclines).

(c) Using your graph from the previous part indicate with arrows whether F and G are increasing, decreasing or constant in each of the regions in the positive quadrant your lines created.

(d) Determine whether the equilibrium with $F = 0$ that you found in the first part is stable under small changess to both F and G (e.g. occasionally a few F appear from a distant population).

(e) If both the populations start at 2000 the system is periodic. For which value of the population of G will the population of F reach its lowest value?

5. (25 points) A group of 1000 students is tested for COVID every week and each is identified as Susceptible, Infected or Resistant. Write S_t, I_t and R_t for the numbers of each after t weeks and assume that initially all 1000 were Susceptible.
Each week $\frac{2}{100}$ of those who are Susceptible become Infected and the rest remain Susceptible. Each week $\frac{8}{10}$ of those who are Infected become Resistant and the rest remain Infected. Each week $\frac{1}{100}$ of those who are Resistant become Susceptible and the rest remain Resistant.

(a) Draw a state diagram describing this system.

(b) Write a matrix equation for S_t, I_t and R_t.

(c) Find the number of resistant students after two weeks.

6. (25 points) Consider the matrix $A = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}$.

(a) Find the square A^2.

(b) Find the two eigenvalues for A.

The columns of the matrix $P = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}$ are eigenvectors of A.

(c) Find the inverse P^{-1}.

(d) Assume that the matrix equation $\begin{bmatrix} X_{t+1} \\ Y_{t+1} \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} X_t \\ Y_t \end{bmatrix}$

holds and $\begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} = \begin{bmatrix} 1000 \\ 0 \end{bmatrix}$.

Find the general solutions for X_t and Y_t.

7. (Extra Credit: You do not need to do this problem.)(25 points)
The linearization of problem 4 about its fixed point yields powers of the matrix

$$M = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

Find the complex eigenvalues and eigenvectors of M.