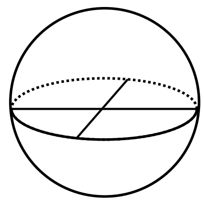
TOPOLOGY PRELIM EXAM: SPRING 2023

(1) Let X be the union of the unit sphere S^2 with two intersecting diameters connecting $(1,0,0)$ with $(-1,0,0)$ and $(0,1,0)$ with $(0,-1,0)$ respectively.



(a) Find a cell decomposition of X. How many cells are there?
(b) Compute $\pi_1(X)$.

(2) Let $S^3 = \{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1\}$ be the unit sphere in $\mathbb{C}^2 = \mathbb{R}^4$. Consider the map

$$g : S^3 \to S^3,\ g(z_1, z_2) = (iz_1, -iz_2).$$

(a) Prove that $g^4 = \mathrm{Id}$ and g generates a free action of $\mathbb{Z}_4$ on S^4.
(b) Compute $\pi_1(S^3/\mathbb{Z}_4)$.

(3) Find the degree of the map $f : S^1 \to S^1$ given by

$$f(x, y) = (y^2 - x^2, 2xy).$$

(4) Let $A = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2\}$ and let
$B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$.
(a) Prove that A and B intersect transversally in $\mathbb{R}^3$.
(b) Prove that the intersection $A \cap B$ is a manifold, and state its dimension.

(5) Let $X = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^3 - z^2 + w^3 = 1\}$.
(a) Prove that X is a manifold and state its dimension.
(b) Calculate (describe mathematically) the tangent spaces of X at a point $(x, y, z, w) \in X$.
(c) Show that $((x, y, z, w), \langle 3y^2, -2x, 3w^2, 2z\rangle)$ is a vector field on X.

(6) Let $\eta = y dx \wedge dy + x dy \wedge dz$ be a 2-form on $\mathbb{R}^3$. Let $f : [0,1] \times [0,1] \to \mathbb{R}^3$ be the map from the unit square defined by $f(s,t) = (s + t, s - t, s^2 + t^2)$. Calculate $\int_{[0,1]\times[0,1]} f^*\eta$.