Qualifying Exam Proposal

[REDACTED]

August 13, 2018

1 Committee and Logistics

1.1 Committee Members

Prof. Adam Jacob

Prof. Andrew Waldron (Chair)

Prof. Joel Hass

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Prof. Steve Carlip

1.2 Logistics

Date Tuesday, August 21, 2018

Time 11AM

Location MSB 3106

2 Proposed Research Talk

2.1 Background and Motivation: Inverse Curvature Flows and the Riemannian Penrose Inequality

The study of Mean Curvature Flow (MCF) arises naturally in Riemannian geometry. Given a closed smooth manifold Σ and a one-parameter family of embeddings $f_t : \Sigma \to M$ into a Riemannian manifold (M, g) with codimension greater than zero, we say Σ_t is a solution to the MCF if

$$\frac{d\Sigma_t}{dt} = -H\nu \tag{1}$$

where $\Sigma_t = f_t(\Sigma)$, ν is the outward-pointing unit normal, and H is the mean curvature of Σ_t in (M, g). MCF can be shown to be the gradient flow for the area of a surface, and it is obvious from (1) that the equilibrium solutions of MCF are surfaces of vanishing mean curvature, i.e. minimal surfaces of (M, g), see [Cho15]. Inverse mean curvature flow (IMCF), whose solution is a one-parameter family of closed surfaces Σ_t in (M, g) satisfying

$$\frac{d\Sigma_t}{dt} = \frac{1}{H}\nu \tag{2}$$

has also garnered considerable interest from geometers in recent decades, although the underlying motivation for the study of IMCF is quite different than the underlying motivation for the study of MCF. Specifically, IMCF has proven to be a powerful tool in the mathematical study of general relativity.

The concept of mass in general relativity has a long and complicated history. One definition involves considering certain spacelike hypersurfaces of 4-dimensional Lorentzian manifolds and thus can be formulated strictly through the viewpoint of Riemannian Geometry. One such type of hypersurface is called an asymptotically Euclidean manifold.

Definition 1. *A 3-dimensional Riemannian Manifold (M, g) is called asymptotically Euclidean if there exists a compact set $K \subset M$ such that*

1. *There exsits a diffeomorphism $y : M \setminus K \to \mathbb{R}^3 \setminus B_1(0)$*
2. *In this coordinate chart, the components of g satisfy*

$$g_{ij} - \delta_{ij} = \mathcal{O}(|y|^{-\alpha}) \tag{3}$$
$$\partial_k g_{ij} = \mathcal{O}(|y|^{-\alpha-1}) \tag{4}$$

for some $\alpha > \frac{1}{2}$.

Furthermore, (M^3, g) *is called Harmonically flat at infinity if there exists a chart satisfy the above conditions where* $g_{ij}(y) = u(y)^4\delta_{ij}$ *for some scalar function* u.

Definition 2. *Given an asymptotically Euclidean 3-manifold* (M^3, g)*, define its ADM Mass* m_{ADM} *to be*

$$m_{ADM} = \frac{1}{16\pi} \lim_{\sigma\to\infty} \int_{S_\sigma} g_{ij,i}\nu_j - g_{ii,j}\nu_j d\mu \tag{5}$$

One might notice that the right-hand side of (5) involves partial derivatives and thus appears to be coordinate-dependent quantity– and indeed it is! However, one can also show that this quantity is invariant over charts which satisfy the decay condition (3). Thus m_{ADM} provides a well-defined notion of total mass of an Asymptotcially Euclidean spacelike hypersurface of a spacetime.

One central conjecture about the ADM Mass of an asymptotically Euclidean manifold was the Riemannian Penrose Inequality (RPI).

Theorem 1 (Riemannian Penrose Inequality)**.** *Let* (M^3, g) *be a complete Riemannian 3-manifold with nonnegative scalar curvature which is harmonically flat at infinity. If* (M^3, g) *admits an outermost minimal surface* Σ*, then the inequality*

$$m_{ADM} \geq \sqrt{\frac{|\Sigma|}{16\pi}} \tag{6}$$

holds, with equality if and only if $(M^3, g) \simeq (\mathbb{R}^3 \setminus 0, g_S)$*, where* g_S *is the spatial Schwarzschild metric given by*

$$g_S = (1 + \frac{m}{2|x|})^4\delta_{ij} \tag{7}$$

Here, the outermost minimal surface is defined as the minimal surface bounding an open subset of M formed by the union of all open subsets of M which are bounded by minimal surfaces. In the context of general relativity, the outermost minimal surface over a constant-time slice of a black hole spacetime corresponds to the black hole's event horizon (Though one must note though that for the Riemannian Penrose Inequality to provide information about the spacetime geometry, the spacelike hypersurface must be totally geodesic within the spacetime, see [BC03]). Thus intuitively the RPI provides a lower bound on a black hole's total mass in terms of its area.

The inequality was originally proven by Huisken and Illamen in [HI01]. IMCF proved to have a property which, given certain long-time existence and convergence properties, would immediately imply the RPI.

Definition 3. *Given a smooth submanifold Sigma of* (M, g)*, its Hawking Mass* m_H *is given by*

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}(1 - \int_\Sigma H^2 d\sigma) \tag{8}$$

Notice that if Σ is the outermost minimal surface of (M, g), then $m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}$. The critical property of IMCF is that the Hawking Mass is non-decreasing along the flow. Furthermore, the Hawking Mass of round spheres at infinity approximates the ADM mass of (M, g) from below. Therefore, if one flows by IMCF starting from an outermost minimal surface $\Sigma_0 = \Sigma$ and the IMCF converges to round spheres at infinity, monotonicty directly implies the RPI in 3 dimensions.

Of course, there are a number of pathologies arising in this flow, most notably that the right-hand side of (2) is undefined whenever Σ_t has a point of vanishing mean curvature. It is for this reason, Huisken and Illamen used a generalization of IMCF. This "weak" IMCF is equivalent to IMCF over flows with non-vanishing mean curvature, but over flows where the IMCF approaches a surface whose mean curvature vanishes somewhere, weak IMCF will "jump past" these problematic regions. For a more in-depth discussion of weak IMCF, see [HI01].

2.2 The Asymptotically Hyperbolic Case

In recent years, another class of Riemannian manifolds known as Asymptotically Hyperbolic (AH) Manifolds have received an increasing amount of interest from physicists and geometers. Our definition of an AH manifold given in [All18] appeals to conformal compactifications.

Definition 4. *Let $\bar{M}$ be a smooth, compact $n+1$-dimensional manifold with boundary ∂M and interior M. Given a Riemmannian metric g on M, the Riemannian manifold (M, g) is said to be conformally compact if there exists a smooth function $\rho \in C^\infty(\bar{M})$ with $\rho^{-1}(0) = \partial M$, $d\rho \neq 0$ on ∂M such that the metric $\rho^2 g$ extends continuously to a non-degenerate metric on M.*

One familiar example of a conformally compact Riemannian manifold is the Poincare Ball $B_2(0)$ equipped with the metric $g = \frac{4}{1-|x|^2}^2 dx_i^2$. Indeed, although the open ball $B_2(0)$ is already geodesically complete with respect to g, conformally rescaling by the defining function $\rho = \frac{1-|x|^2}{2}$, we can extend $\bar{g} = \rho^2 g = \delta$ to the Euclidean metric over $\bar{B}_2(0)$. In fact, this is the prototypical example of an AH manifold.

Definition 5. *A conformally compact Riemannian manifold (M^3, g) is called asymptotically hyperbolic if*

1. *The conformal boundary of $(\bar{M}, \bar{g})$ is the standard sphere (S^2, g_0).*
2. *The defining function ρ satisfies*
$$|d\rho|^2_{\bar{g}} = 1 \tag{9}$$
Over M for $\bar{g} = \rho^2 g$.
3. *We may write in a collar neighborhood of the conformal boundary*
$$g = sinh^{-2}(\rho)(d\rho^2 + g_\rho) \tag{10}$$
With
$$g_\rho = g_0 + \frac{r^n}{n} h + O(\rho^{n+1}) \tag{11}$$
Where h is a symmetric 2-tensor on S^{n-1} and g_ρ is the metric induced over a level set of ρ.

One may show that the sectional curvatures of g approach $-|d\rho|_g^2 = -1$ near the conformal boundary, hence the name.
Wang in [Wan01] uses the Conformal Boundary $(\mathbb{S}^2, g_{\mathbb{S}^2})$ to define the mass of an AH Manifold.

Definition 6. *The Wang Mass of an asymptotically hyperbolic manifold is defined as*

$$m = ((\int_{\mathbb{S}^2} tr_{g_{\mathbb{S}^2}} h d\mu_{g_{\mathbb{S}^2}})^2 - (\int_{\mathbb{S}^2} x_i tr_{g_{\mathbb{S}^2}} h d\mu_{g_{\mathbb{S}^2}})^2)^{\frac{1}{2}} \tag{12}$$

Naturally, one would like to find an analog of the RPI for the Wang Mass, and this has been a central focus in the study of AH Manifolds in recent years.

The conjectured Riemannian Penrose Inequality for AH Manifolds is as follows:

Conjecture 1 (Riemannian Penrose Inequality for Asymptotically Hyperbolic Manifolds)**.** *Let (M^3, g) be an asymptotically hyperbolic 3-manifold with $R \geq -6$ and an outermost sphere with Σ_0 with $H = 2$. Then*

$$m \geq \sqrt{\frac{|\Sigma|}{16\pi}} \tag{13}$$

With equality if and only if (M^3, g) is isometric to the Anti de Sitter-Schwarzschild space outside Σ_0.

Initially, the research community hoped that Huisken and Illamen's method of exploiting monotonicity of the Hawking Mass to obtain the RPI in the AE setting could be translated to the AH setting. Although IMCF mantains its monotonicty property over AH manifolds for the modified Hawking Mass $m_H(\Sigma) = \frac{|\Sigma|}{8\pi}(1 - \int_\Sigma (H^2 - 4) d\sigma$, a result by Neves in [Nev10] showed that the convergence properties of IMCF on an AH manifold are in general insufficient to allow for an identical argument.

Theorem 2. *There is an asymptotically hyperbolic 3-manifold (M, g) with scalar curvature -6 and for which its boundary Σ_0 is an outer-minimizing $H(\Sigma_0) = 2$ sphere such that the solution of IMCF with initial condition Σ_0 does not converge to round spheres at infinity.*

This construction begins with an initial surface in Anti deSitter Schwarzschild space whose IMCF does not converge to round spheres at infinity, in the sense that the Gauss curvatures of the Σ_t do not approach 1 as $t \to \infty$. Next, Neves constructs a new metric over the region of ADS-Schwarzschild foliated by Σ_t using a scalar function u satisfying $u|_{\Sigma_0} = \frac{H_{\Sigma_0}}{2}$. Given that g_{ADSS} can be written

$$g_{ADSS} = \frac{dt^2}{H^2} + g_t \tag{14}$$

where the t coordinate is given by the flow parameter and g_t is the metric induced on Σ_t. He defines a new metric g by

$$\hat{g} = \frac{u^2 dt^2}{H^2} + g_t \tag{15}$$

Note that since the t direction is normal to Σ_t, one can readily compute the new mean curvature as $\hat{H}(\Sigma_t) = \frac{H(\Sigma_t)}{2}$ and $\hat{\nu} = \frac{\nu}{u}$. This implies both that $\hat{H}(\Sigma_0) = 2$ and that $\frac{\hat{\nu}}{\hat{H}} = \frac{\nu}{H}$. So the initial surface is an outermost $H = 2$ surface with respect to $\hat{g}$ and the solution to the IMCF with respect to $\hat{g}$ is the solution to the IMCF with respect to g. Since both ambient metrics induce the same metric on Σ_t, we know by the Theorem Egregium of Gauss that the Gauss curvatures of Σ_t with respect to the re-normalized metric $|\Sigma_t|\hat{g}$ must not converge to 1, and the limit of the Hawking Mass overestimates the Wang Mass of the foliated region, and thus the monotonicity argument in its current incarnation is inapplicable.

2.3 Thesis Goals and Future Work

The ultimate goal of the thesis would be to find some other means of proving the RPI for AH Manifolds. Though the route of utilizing the properties of IMCF appears less straightforward than expected, it could very well be that some sort of geometric flow, be it intrinsic or extrinsic, holds the key to providing a proof of this conjecture.

2.3.1 Uniqueness of Hawking Mass Monotonicty along IMCF

First, it is clear from the previous section that the convergence properties of IMCF on AH manifolds are not well-understood. As a stepping stone toward a more complete picture of these properties, I plan to begin the project by studying IMCF in $\mathbb{H}^{n+1}$, a topic which has seen progress in recent years. In [Ger11], Gerhardt showed that in $\mathbb{H}^{n+1}$ if one takes the initial hypersurface to be star-shaped with respect to a point, then the IMCF will exist for all time, and its leaves will both become strongly convex and have principal curvatures approaching 1 exponentially fast. It should be noted that, like in Neves's counterexample, the Gauss curvature will in general not converge to 1 with respect to the renormalized induced metric $\hat{g}_t = |\Sigma| g_t$, as shown by Hung and Wang in [HW15]

Another note of interest on Gerhardt's work is that his results generalize to a larger family of curvature flows in $\mathbb{H}^{n+1}$: he proved the above result for any curvature flow of the form

$$\frac{d\Sigma}{dt} = -\Phi(F)\nu \tag{16}$$

where F is a degree 1 polynomial of the principal curvatures of Σ_t, i.e. $F(\Sigma) = c_0 + \sum_i c_i k_i$ where k_i are the eigenvalues of the Second Fundamental Form of Σ, and Φ is any function of F with $\dot{\Phi} > 0$ and $\ddot{\Phi} < 0$.

One potentially interesting question is whether or not there exist any other curvature flows in this family which carry the monotonicity property for the Hawking Mass, and if so, whether or not such a flow maintains this monotonicity property on any AH manifold. If there does exist another inverse curvature flow over which the Hawking Mass is monotone, a natural next step would be to check if the existing counterexample to convergence to round spheres at infinity still applies to it. Indeed, some alternative flow might be necessary to translate Huisken and Illamen's original argument to the AH setting.

2.3.2 IMCF over Non Star-Shaped Surfaces

I also hope to understand which of these results may be applied to a more general class of hypersurfaces or to a different ambient AH Manifold. Some progress was made on this former by Allen in [All18], where he showed long-time existence for IMCF for a family of non-compact hypersurfaces in $\mathbb{H}^{n+1}$, namely bounded graphs over horospheres. Some progress on the latter was made by Lu in [Lu17] managed to show longtime existence, smoothness, and convergence of principal curvatures to 1 for IMCF for star-shaped initial hypersurfaces in ADS-Schwarzschild space.

In [Ger11] and [Lu17], the initial hypersurface was taken to be star-shaped with respect to some point p so that it may be viewed as a graph in the radial coordinate with respect to geodesic normal coordinates centered at p. While in the short term I would still limit my study to graphical hypersurfaces of H^{n+1} and ADS-Schwarzschild, I would like to understand which of these convergence properties hold over graphical initial surfaces which are not necessarily star-shaped. Thus I plan to study IMCF in $\mathbf{H}^{n+1}$ over some graphical non-star-shaped initial surfaces, possibly the DeLaunay surfaces given in [Kor+92]. Regardless of whether or not this proves a stepping stone in a proof of the RPI in the AH setting, it will at the very least improve the understanding IMCF in $\mathbb{H}^{n+1}$

2.3.3 Curvature Bounds for IMCF over all AH Manifolds

The underlying ambition is still a proof of the RPI for asymptotically hyperbolic manifolds, and as such, I will eventually transition to a study of IMCF on all such spaces, or at least a larger subset of them. Natural questions arise about long-time existence, smoothness, and curvature estimates of solutions over other AH spaces. For example, Gerhardt obtains that over star-shaped initial hypersurfaces with positive mean curvature that the mean curvature is bounded away from 0 for all times. One then might ask if this is true for any star-shaped surface in an AH space. This would imply that the weak formulation of IMCF presented by Huisken and Illamen would not be necessary when considering such initial surfaces. Some analytic aspects of studying IMCF may prove less tractable when one is not working with a specific background metric. However, it is also possible that the geometry of these manifolds may provide some insight into various other properties of the flow.

References

[Kor+92] N. Korevaar et al. "Constant Mean Curvature Surfaces in Hyperbolic Space". In: *American Journal of Mathematics* 114.1 (1992), pp. 1–43.

[HI01] Gerhard Huisken and Tom Illamen. "The Inverse Mean Curvature Flow and the Riemannian Penrose Inequality". In: *J. Differential Geometry* 59.3 (2001), pp. 353–437. DOI: `doi:10.4310/jdg/1090349447`.

[Wan01] Xiadong Wang. "The Mass of Asymptotically Hyperbolic Manifolds". In: *J. Differential Geometry* 57.2 (2001), pp. 273–299. DOI: `doi:10.4310/jdg/1090348112`.

[BC03] Hubert Bray and Piotr Chrusciel. "The Penrose Inequality". In: *The Einstein Equations and the Large-Scale Behavior of Gravitational Fields* (2003), pp. 39–70.

[Nev10] Andre Neves. "Insufficient Convergence of Inverse Mean Curvature Flow on Asymptotically Hyperbolic Manifolds". In: *J. Differential Geometry* 84.1 (2010), pp. 191–229. DOI: `doi:10.4310/jdg/1271271798`.

[Ger11] Claus Gerhardt. "Inverse Curvature Flows in Hyperbolic Space". In: *J. Differential Geometry* 89.3 (2011), pp. 487–527. DOI: `doi:10.4310/jdg/1335207376`.

[Cho15] Otis Chodosh. "Brian White - Mean Curvature Flow (math 258) Lecture Notes". In: 2015.

[HW15] Pei-Ken Hung and Mu-Tao Wang. "Inverse mean curvature flows in the hyperbolic 3-space revisited". In: *Calculus of Variations and Partial Differential Equations* 54.1 (2015), pp. 119–126. DOI: `doi.org/10.1007/s00526-014-0780-3`.

[Lu17] Siyuan Lu. "The Inverse Mean Curvature Flow and the Riemannian Penrose Inequality". In: *Comm. Anal. Geometry* (2017).

[All18] Brian Allen. "ODE Maximum Principle at Infinity and Non-Compact Solutions to IMCF in Hyperbolic Space". In: (2018).

3 Proposed Qualifying Exam Syllabus

1. **Analysis**

   - Banach Spaces
     - Bounded and compact mappings
     - Spaces of continuous functions
     - Examples of Banach spaces
   - Measure Theory
     - Pre-measures and outer measures
     - Lebesgue Measure, Atomic Measure
     - Monotone Convergence Theorem, Dominated Convergence Theorem, Fatou's Lemma
     - Absolutely continuous measures, Radon-Nikodym Theorem
   - Fourier Analysis
     - Fourier bases of L^2, Fourier characterization of H^s
     - The Fourier Transform on Schwartz functions
     - Riemann-Lebesque Lemma, Plancherel's Theorem

   Reference Text(s): *Applied Analysis*, J. Hunter, B. Nachtergaele. *Measure Theory: Second Edition*, D. Cohn.

2. **Functional Analysis**

   - Spectral Theory for Hilbert Spaces
     - Resolvents, Point and Residual Spectra
     - Spectral Theorem for Compact, Self-Adjoint Operators
     - Eigenvalue perturbations
   - Characterization of Dual Spaces
     - Banach-Alaglou Theorem
     - Riesz Representation Theorem
     - Lax-Milgram Theorem
   - Miscellaneous Theorems
     - Open Mapping Theorem

    - Closed Graph Theorem
    - Hahn-Banach Theorem

  Reference Text(s): *Functional Analysis, Sobolev Spaces, and Partial Differential Equations*, H. Brezis.

3. **Partial Differential Equations**

  - Sobolev Spaces
    - Definition, Elementary Properties
    - Sobolev Inequalities
    - Embedding Theorems
    - Trace and Extension Theorems
  - Existence and Regularity Theory for Linear, Uniformly Elliptic PDE
    - Existence of Weak Solutions for Homogeneous Boundary Value Problems
    - A Priori Estimates
    - Interior and Boundary Regularity of Weak Solutions
    - Strong and Weak Maximum Principles, Hopf Lemma
    - Existence and Regularity for Non-homogeneuous BVP
  - Calculus of Variations
    - First Variation, Euler-Lagrange Equations
    - Second Variation
    - Existence of Minimizers

  Reference Text: *Partial Differential Equations: Second Edition*, L. Evans. *Elliptic Partial Differential Equations*, Q. Han and F. Lin.

4. **Smooth Manifold Theory**

  - Differentiable Manifolds
    - Definition of Smooth Manifold
    - Paracompactness, Orientability
    - Smooth Maps, Immersions, Embeddings
    - Submanifolds, Regular Level Set Theorem

- Tangent and Cotangent Bundles
  - Definition
  - Push-forwards and Pull-backs
  - Lie Derivative
  - Higher-Order Tensor Fields, Exterior Derivative
- Lie Groups and Lie Algebras
  - Definition of a Lie Group, examples
  - Matrix Groups
  - Lie Algebra of a Lie Group, Adjoint Representation, examples
  - Left- and Right-Invariant Vector Fields
  - Representations of $U(1), SU(2)$
  - Complexifications of a Lie Algebra
  - Representations of $sl(2, \mathbb{C})$, Weights and Weight Vectors, Roots and Root Vectors
  - Semisimple Lie Algebras
  - Compact Lie Groups and Reductive Lie Algebras

Reference Text(s): *Introduction to Smooth Manifolds*, L. Tu. *Representation Theory: A First Course*, W. Fulton and J. Harris.

5. **Riemannian Geometry**

- Riemannian Metrics, Connections
  - Definition of Riemannian Metric, Existence, Examples
  - Levi-Civita Connections
  - Parallel Transport, Geodesics, Exponential Map
- Curvature
  - Riemann Curvature Tensor, Bianchi Identities, Ricci and Scalar Curvature
  - Sectional Curvature
  - Hyperbolic Space– Isometries, Geodesics
- Hypersurface Geometry
  - Gauss Curvature, Gauss-Bonnet Theorem

  - Second Fundamental Form, Mean Curvature, Totally Geodesic Submanifolds
  - Minimal Surfaces, Minimal Graph Equation

Reference Text: *Riemannian Geometry*, M. Do Carmo.