ALGEBRA PRELIM EXAM: SPRING 2023

(1) Find a number larger than one hundred so that every group of that order has exactly two elements of order three.

(2) Find two fields of the same finite degree over their prime fields with nonisomorphic Galois groups. Hint: The prime fields need not be the same.

(3) Show that there is a nonabelian group of order 55 but none of order 77.

(4) Consider the free $\mathbb{Z}$-module $M = \mathbb{Z} \times \mathbb{Z}$. Show $\{(a, b), (c, d)\}$ is a basis of M if and only if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{Z})$.

(5) Let R be a commutative unital ring and M a cyclic R-module. Show that the tensor algebra $T(M)$ is commutative.

(6) Let $R = \mathbb{F}_7[x]$ and $I = (x^3 + 5x^2 + x + 5)$ and $\overline{R} = R/I$.
   (a) Is $\overline{x^2 - 4} := x^2 - 4 + I \in \overline{R}$ a unit of $\overline{R}$? If yes, provide a multiplicative inverse. If no, explain why not.
   (b) Is $\overline{x} := x + I \in \overline{R}$ a unit of $\overline{R}$? If yes, provide a multiplicative inverse. If no, explain why not.