ANALYSIS PRELIM EXAM: SPRING 2023

(1) Recall that $C_0(\mathbb{R})$ is the closure, in the sup norm, of the space of continuous functions on $\mathbb{R}$ with compact support. A family of functions $\mathcal{F} \subset C_0(\mathbb{R})$ is said to be tight if for every $\epsilon > 0$ there exists $R > 0$ such that $|f(x)| < \epsilon$ for all $x \in \mathbb{R}$ with $|x| \geq R$, and all $f \in \mathcal{F}$. Prove that $\mathcal{F} \subset C_0(\mathbb{R})$ is precompact in $C_0(\mathbb{R})$ if it is bounded, equicontinuous, and tight.

(2) For $1 \leq p < \infty$, recall the normer linear space ℓ^p of real sequences $a = \{a_1, a_2, a_3, ...\}$ that have finite norm $||a||_p$, where

$$||a||_p := \left(\sum_{k=1}^{\infty} |a_k|^p \right)^{\frac{1}{p}}.$$

Prove that ℓ^p is complete, and thus a Banach Space.

(3) Let $(\mathcal{B}, ||\cdot||_{\mathcal{B}})$ be a Banach space and V be a closed linear subspace. The quotient $\mathcal{Q} := \mathcal{B}/V$ is defined to be the set of equivalence classes of the relation $\sim$, where $x \sim y$ if and only if $(x - y) \in V$. There exists a natural surjection $\phi : \mathcal{B} \to \mathcal{Q}$. For any $y \in \mathcal{Q}$, since $y = \phi(x)$ for some x we consider the norm

$$||y||_{\mathcal{Q}} = ||\phi(x)||_{\mathcal{Q}} := \inf_{v \in V} ||x - v||_{\mathcal{B}}.$$

(note unlike the Hilbert space case this infimum may not be realized)

(a) Prove the norm $||y||_{\mathcal{Q}}$ is well defined independent of the choice of x.

(b) Prove the operator norm of ϕ is 1.

(4) Suppose that $f \in L^1(\mathbb{R})$ satisfies

$$\limsup_{h \to 0} \int_{\mathbb{R}} \left| \frac{f(x+h) - f(x)}{h} \right| dx = 0\,.$$

Show that $f = 0$ almost everywhere. (*Hint.* Show that $\int_{\mathbb{R}} |f(y)|dy = 0$.)

(5) Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $f \in L^p(\mathbb{R}^3)$ and $g \in L^q(\mathbb{R}^3)$. Show that $f * g$ is continuous on $\mathbb{R}^3$.

(6) Suppose that $u \in L^2(\mathbb{R})$ such that $\left(1 + |\xi|^{\frac{1}{2}}\right) \hat{u}(\xi) \in L^2(\mathbb{R})$, where $\hat{u}$ denotes the Fourier transform of u. Show that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(x+y) - u(x)|^2}{|y|^2} dxdy \leq C \int_{\mathbb{R}} |\xi|\, |\hat{u}(\xi)|^2 d\xi\,,$$

where $C > 0$ denotes a universal constant. (*Hint.* Use the Fourier transform of the translation operator.)