JUNE 2025 ANALYSIS PRELIM EXAM

Name:

Student ID:

This exam has 6 problems. Each problem is worth 10 points. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Please write your solutions in this exam booklet. If you need an extra page for part of your solution that you want graded, write a note in the pages within the test booklet for that problem, and clearly write the problem number and your student ID on the extra page. Append any extra pages you want graded to the end of your exam.

(1) Suppose that $F : X \to \mathbb{R}$ is a bounded linear functional on a real normed linear space X with $\|F\| = 1$. Prove that $\operatorname{dist}(x, \ker F) = |F(x)|$ for every $x \in X$.

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(2) Let $C_0[0,1]$ denote the Banach space of continuous functions $f : [0,1] \to \mathbb{R}$ with $f(0) = f(1) = 0$, equipped with the sup-norm $\|f\| = \sup_{x\in[0,1]} |f(x)|$. Let $\mathcal{F} \subset C_0[0,1]$ be the subset of Lipschitz continuous functions, and $\mathcal{G} \subset C_0[0,1]$ be the subset of Lipschitz continuous functions with Lipschitz constant less than one i.e., $g \in \mathcal{G}$ if

$$\sup_{x\neq y\in[0,1]} \frac{|g(x) - g(y)|}{|x - y|} < 1.$$

(a) Is $\mathcal{F}$ relatively compact in $C_0[0,1]$?

(b) Is $\mathcal{G}$ relatively compact in $C_0[0,1]$?

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(3) Let $\ell^2(\mathbb{N})$ be the Hilbert space of square-summable sequences $x = (x_n)_{n=1}^{\infty}$, where $x_n \in \mathbb{R}$, with norm

$$\|x\| = \left(\sum_{n=1}^{\infty} x_n^2 \right)^{1/2}$$

and closed unit ball $B = \{x \in \ell^2(\mathbb{N}) : \|x\| \leq 1\}$. Define $T : B \to \ell^2(\mathbb{N})$ by

$$T(x_1, x_2, x_3, x_4, \dots) = (\sqrt{1 - \|x\|^2}, x_1, x_2, x_3, \dots).$$

Prove that T is a continuous function from B into B that has no fixed point (i.e., a point $x \in B$ such that $Tx = x$).

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(4) Assume that $f \in L^1(\mathbb{R})$, show that the Fourier transform $\hat{f}$ is uniformly continuous.

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(5) Let (Ω, Σ, μ) be a measure space and let $\{A_n\}$ be an increasing sequence of measurable sets, increasing means that $A_n \subset A_{n+1}$. Show that

$$\lim_{n\to\infty} \mu(A_n) = \mu(\cup_n A_n).$$

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(6) Consider the operator $L : L^2(\mathbb{R}) \to L^2(\mathbb{R})$,

$$(Lf)(x) := h(x)f(x),$$

where $h(x) = x$ for $x \in (-1, 1)$, $h(x) = 1$ for $x \geq 1$ and $h(x) = -1$ for $x \leq -1$. Show that L is a bounded operator. Find the spectrum and the point spectrum of L.

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