JUNE 2025 TOPOLOGY PRELIM EXAM

Name:

Student ID:

This exam has 6 problems. Each problem is worth 10 points. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Please write your solutions in this exam booklet. If you need an extra page for part of your solution that you want graded, append the extra page to the end of your exam, write a note saying continued on extra page in the page within the test booklet for that problem, and clearly write the problem number and your student ID on the extra page.

(1) Consider the subsets of $\mathbb{R}^3$

$$X := \{(x,y,z) \in \mathbb{R}^3 : (\sqrt{x^2+y^2} - 10)^2 + z^2 = 1\}, \quad S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 100\}$$

(a) Show that the intersection $S \cap X$ is transverse.

(b) Consider the map $f : X \longrightarrow \mathbb{R}$ given by $f(x,y,z) = z$. Find its critical points and critical values.

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(2) Consider the 3-dimensional manifold $S^1 \times S^2 = \{(\theta, x, y, z) \in \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ and the 1-form $\alpha = zd\theta + (xdy - ydx) \in \Omega^1(S^1 \times S^2)$.

(a) Compute $d\alpha \in \Omega^2(S^1 \times S^2)$ and its restriction to the tangent space at the point $(\pi, 1, 0, 0) \in S^1 \times S^2$.

(b) Show that $\alpha \wedge d\alpha$ is a nowhere vanishing 3-form on $S^1 \times S^2$.

(c) Consider curves $\gamma_d := \{\theta = d, z = 0\}$ for $d \in [0, 2\pi)$, a coordinate on S^1. For which values of d does there exist a point in γ_d where α vanishes on its tangent space? (Justify your answer.)

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(3) Denote the real $n \times n$ matrices by $M_n(\mathbb{R})$, and note we can identify this set with $\mathbb{R}^{n^2}$ to give it the Euclidean topology. Consider the subset of $M_n(\mathbb{R})$ given by

$$X_n = \{A \in M_n(\mathbb{R}) : AA^t = I\} \subseteq \mathbb{R}^{n^2}.$$

(a) Prove that X_n is a smooth manifold and compute its dimension.

(b) Show that the tangent space of X_n at the point $A = \mathrm{Id}$ coincides with the vector subspace of skew-symmetric matrices in $M_n(\mathbb{R})$.

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(4) Recall that the join $X * Y$ of two topological spaces X and Y is the quotient space of $X \times Y \times [0,1]$ under the equivalence relation $(x, y_1, 0) \sim (x, y_2, 0)$ for all $y_1, y_2 \in Y$, $x \in X$, and $(x_1, y, 1) \sim (x_2, y, 1)$ for all $x_1, x_2 \in X$, $y \in Y$ (and otherwise only $(x, y, t) \sim (x, y, t)$ for $t \in (0, 1)$). Prove that if X and Y are non-empty and X is path connected, then $X * Y$ is simply connected.

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(5) Let X be the cell complex built from one 0-cell $e^0 = \{0\}$, one 1-cell e^1 (attached to the 0-cell along its endpoints by the constant map), and one 2-cell e^2 attached via the map $\Phi_n : S^1 = \partial e^2 \to S^1 = X^1$ defined by

$$\Phi_n(\cos(\theta), \sin(\theta)) = (\cos(n\theta), \sin(n\theta)).$$

(Here we identify the 1-skeleton X^1 with S^1 in the standard way $S^1 \cong [0, 1]/0 \sim 1$.)

Give the universal cover $\tilde{X}$ and covering map $p : \tilde{X} \to X$, (specifying $\tilde{X}$ by a cell complex with attaching maps) and verify p is actually a covering map and that $\tilde{X}$ satisfies the necessary properties of the universal cover.

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(6) We can define

$$S^{2n+1} = \{(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \mid |z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = 1\}$$

(where $z_j = x_j + iy_j$ and $|z_j|^2 = x_j^2 + y_j^2$) and

$$\mathbb{C}\mathrm{P}^n = \{[z_0 : z_1 : \cdots : z_n] \mid (z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$$
$$\text{where } [z_0 : z_1 : \cdots : z_n] = [\lambda z_0 : \lambda z_1 : \cdots : \lambda z_n], \lambda \in \mathbb{C} \setminus \{0\}\}.$$

(a) Show that the quotient map $p : S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ given by $p(z_0, z_1, \ldots, z_n) = [z_0 : z_1 : \cdots : z_n]$ is a locally trivial fiber bundle with fiber homeomorphic to S^1.

(b) Prove that for all $n \geq 1$, $\pi_k(S^{2n+1}) \cong \pi_k(\mathbb{C}\mathrm{P}^n)$ for all $k \geq 3$ and $\pi_2(\mathbb{C}\mathrm{P}^n) \cong \mathbb{Z}$.

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