JUNE 2025 ALGEBRA PRELIM EXAM

Name:

Student ID:

This exam has 6 problems. Each problem is worth 10 points. You should give full proofs or explanations of your solutions. Remember to state or cite theorems that you use in your solutions.

Please write your solutions in this exam booklet. If you need an extra page for part of your solution that you want graded, write a note in the pages within the test booklet for that problem, and clearly write the problem number and your student ID on the extra page. Append any extra pages you want graded to the end of your exam.

(1) Let G be a group of order $2pq$ where p, q are odd primes (not necessarily distinct). Show that G must be solvable.

.

(2) Let F be a field and let $K = \bar{F}$ be an algebraic closure of F. Let $f(X) \in K[X]$ be a nonzero polynomial. Show that there exists a nonzero polynomial $g(X) \in F[X]$ such that $g(X) = f(X) \cdot h(X)$ for some $h(X) \in K[X]$.

.

(3) Let $f(X)$ be an irreducible polynomial in $\mathbb{Q}[X]$ with both real and non-real roots. Show that the Galois group of $f(X)$ over $\mathbb{Q}$ is non-abelian.

.

(4) (a) How many **ring** homomorphisms are there $f : \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \to \mathbb{Z}/8\mathbb{Z}$?
Why? Explain.

(b) How many $\mathbb{Z}$-**module** homomorphisms are there $g : \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \to \mathbb{Z}/8\mathbb{Z}$? Why? Explain.

.

(5) Let $\mathbb{F}$ be a field. Explain why $\mathbb{F}(x)$ is not a free $\mathbb{F}[x]$-module.

.

(6) Let $R = \mathbb{Z}[y]$ and consider the ideal $J = (3, y)$ which is thus also an R-module. Consider the element $3 \otimes y - y \otimes 3 \in J \otimes_R J$. Prove this element is NOT zero.

.

.