TOPOLOGY PRELIM EXAM

(1) Let $f : X \to Y$ be a smooth map between compact smooth manifolds of the same dimension. Suppose y_0 is a regular value of f and the number of points in the preimage of y_0 is one, $|f^{-1}(y_0)| = 1$ (i.e. $f^{-1}(y_0) = \{x_0\}$).

(a) Can there exist a point $y \in Y$ with $f^{-1}(y) = \emptyset$? (i.e. $|f^{-1}(y)| = 0$) Prove it cannot exist or give an example where it does exist.

(b) Can there exist a point $y \in Y$ with $|f^{-1}(y))| = 2$? (i.e. $f^{-1}(y) = \{x_1, x_2\}$) Prove it cannot exist or give an example where it does exist.

(2) Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be the smooth map $f(u, v) = (u, 2u + 3v, v)$ and let $H \subset \mathbb{R}^3$ be the hyperboloid $H = \{(x, y, z) \mid x^2 + y^2 - z^2 = 1\}$

(a) Is f transverse to H? Prove or disprove.

(b) Is $f^{-1}(H)$ a submanifold of $\mathbb{R}^2$? If so, what is its dimension?

(3) Let $D^2 = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 \leq 1\}$ be the closed unit disk. Let $f : D^2 \to \mathbb{R}^3$ be the map

$$f(u, v) = (u, v, 1 - u^2 - v^2).$$

Let λ be the 1-form on $\mathbb{R}^3$ given by

$$\lambda = (xyz + ze^x - y)\, dx + (x + z^2)\, dy + 3xy\, dz.$$

Find the value of the integral

$$\int_{D^2} f^*(d\lambda).$$

Here we orient D^2 as a subset of $\mathbb{R}^2$ with the standard orientation on $\mathbb{R}^2$ (∂_u, ∂_v).

(4) (a) Is there a covering map of a surface of genus two where the domain is homeomorphic to a surface of genus four? If yes, describe such a covering map and give its degree. If not, prove why it cannot exist.

(b) Is there a covering map of a surface of genus three where the domain is homeomorphic to a surface of genus four? If yes, describe such a covering map and give its degree. If not, prove why it cannot exist.

(5) For topological spaces X and Y and base points $x_0 \in X$ and $y_0 \in Y$, prove that the following groups are isomorphic

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

Hint: construct the group homomorphism and its inverse. Make sure you justify everything is well-defined when necessary.

(6) Let T be a torus with an open disk removed, and let $p \in T$ be a point on the boundary.

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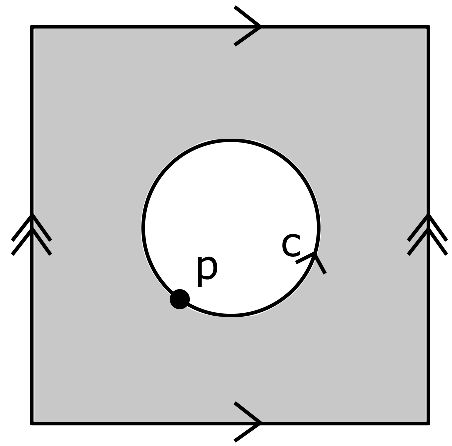

(This figure represents the polygonal representation of the torus).

(a) Prove that $\pi_1(T, p)$ is generated by two based loops a and b, and state any relations. Draw the based loops a and b in the figure above (or your own copy of the figure).

(b) Let c be a loop in the boundary of T that starts and ends at p and goes once around the boundary of T. Express c as a word in $\pi_1(T, p)$ in terms of a and b. Make sure you indicate orientations on a, b and c using arrows.